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RECURSIVE WELL-FOUNDED ORDERINGS

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1. Introduction

Let $W(\alpha)$ denote the set of Gödel numbers of recursive well-orderings of natural numbers of ordinal less than α. The many-one degrees of the sets $W(\alpha)$ for $\alpha < \omega^\omega$ have been completely determined by the accumulated work of Kreisel, Shoenfield, and Wang [3], Liu [4] and [5], and Hay, Manaster and Rosenstein [1]. In this paper, we extend the results on many-one degrees of $W(\alpha)$ to all recursive ordinals α. As a tool, we first investigate the many-one degrees of $WF(\alpha)$ for $\alpha < \omega_1$ where $WF(\alpha)$ is the set of Gödel numbers of recursive well-founded partial orderings of natural numbers of rank less than α and ω_1 is the first non-recursive ordinal. We obtain the many-one degrees of $WF(\alpha)$ and $W(\alpha)$ as in Tables 1 and 2, where $r(\omega \cdot \beta + n) = \omega \cdot \beta + 2n$, $n < \omega$.

Table 1

α	$\deg_m WF(\alpha)$
$k \quad (0 < k < \omega)$	a_2
ω	$a_{2,2}$
$\omega \cdot \beta \quad (1 < \beta < \omega_1)$	$e_{r(\beta)}$
$\omega \cdot \beta + k \quad (1 \leq \beta < \omega_1, 0 < k < \omega)$	$a_{r(\beta)+1}$

Table 2

α	$\deg_m W(\alpha)$
$\omega^\beta (1 < \beta < \omega_1)$	$e_{r(\beta)}$
$\omega^\beta < \alpha < \omega^\beta \cdot 2 \quad (1 \leq \beta < \omega_1)$	$a_{r(\beta)+1}$
$\omega^\beta \cdot (p+1) \quad (1 \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p}$
$\omega^\beta \cdot (p+1) < \alpha < \omega^\beta \cdot (p+2)$ $(1 \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p+1}$

The precise definitions of notations in the above tables will appear in later sections. We conjecture that the two "$\leq$" under $\deg_m W(\alpha)$ of Table 2 can be "$=$".[1] Spector [9] has announced part of the above results on $\deg_m W(\alpha)$.

[1] [1] has proved that "$=$" holds for the case that β is finite.

This paper is organized as follows. Sections 2–7 for the study of $\deg_m$ WF(α) for $\alpha<\omega_1$. Sections 8–13 for the investigation of $\deg_m$ W(α) for $\alpha<\omega_1$. Section 14 for a summary on many-one degrees of WF(α), W(α) and 0(α) where 0(α) is the set of notations for ordinals less than α.

2. The set WF

A two-place predicate $R(x, y)$ is said to be a well-founded relation iff $\{(x, y) \mid R(x, y)\}$ well-founds the set

$$\mathscr{F}(R)=\{x \mid (\exists y)(R(x, y) \vee R(y, x))\}.$$

The set WF will consist of all Gödel numbers f of general recursive functions $f(x, y)$ such that the predicate $f(x, y)=0$ is a well-founded relation. Thus $f \in$ WF iff f satisfies the following conditions WF1–WF5, where α is a function variable.

WF1. $(x)(y)(\exists z)\, T_2(f, x, y, z)$.

WF2. $(x)(y)[\{f\}(x, y)=0 \rightarrow \{f\}(x, x)=0 \;\&\; \{f\}(y, y)=0]$.

WF3. $(x)(y)[\{f\}(x, y)=0 \;\&\; \{f\}(y, x)=0 \rightarrow x=y]$.

WF4. $(x)(y)(z)[\{f\}(x, y)=0 \;\&\; \{f\}(y, z)=0 \rightarrow \{f\}(x, z)=0]$.

WF5. $(\alpha)(\exists x)[\{f\}(\alpha(x+1), \alpha(x)) \neq 0 \vee \alpha(x+1)=\alpha(x)]$.

Let P be the set of natural numbers f which satisfy WF1-WF4, i.e., the Gödel numbers of recursive partial orderings. It is clear that P is π_2^0 and WF is π_1^1. We have $W \subseteq$ WF, $W(\alpha) \subseteq$ WF(α), $L \subseteq P$. As in the proof of Theorem 1 of [8], it can be readily shown that WF is complete π_1^1. We shall investigate the many-one degrees (m-degrees) of WF(α) for $\alpha<\omega_1$ in the following five sections.

3. Classification of the sets WF(α) for α < ω · ω in the arithmetical hierarchy

We write $x \leqslant_e y$ for $\{e\}(x, y)=0$ and $x <_e y$ for $x \leqslant_e y \;\&\; x \neq y$ and note that they are Σ_1 predicates.

Proposition 3.1. *$e \in$ WF(k) is a π_2 predicate of e, k.*

Proof. $e \in \mathrm{WF}(k) \leftrightarrow e \in p \;\&\; (x)[\neg (x)_1 <_e (x)_2 <_e \cdots <_e (x)_k]$.

Proposition 3.2. WF(ω) *is a Δ_3 set.*

Proof. $e \in \mathrm{WF}(\omega) \leftrightarrow (\exists\, k)\; e \in \mathrm{WF}(k)$

$$\leftrightarrow e \in p \;\&\; (\exists\, k)\,(x)[\neg(x)_1 <_e (x)_2 <_e \ldots <_e (x)_k].$$

Now we define recursive functions δ_d, δ_u and δ_b so that[2]

$$\begin{aligned}
\{\delta_d(e, m)\}(x, y) &\simeq \{e\}(x, y) && \text{if} \quad y <_e m,\\
&= 1 && \text{otherwise},\\
\{\delta_u(e, n)\}(x, y) &\simeq \{e\}(x, y) && \text{if} \quad n \leqslant_e x,\\
&= 1 && \text{otherwise}.\\
\{\delta_b(e, m, n)\}(x, y) &\simeq \{e\}(x, y) && \text{if} \quad m \leqslant_e x \;\&\; y <_e n,\\
&= 1 && \text{otherwise}.
\end{aligned}$$

Note that all $\leqslant_{\delta_d(e,m)}$, $\leqslant_{\sigma_u(e,n)}$, $\leqslant_{\delta_b(e,m,n)}$ are certain subrelations of $\leqslant_e$.

Proposition 3.3. *$e \in \mathrm{WF}(\omega + k)$ is π_3 (as a predicate of e, k or fix k as a predicate of e).*

Proof.

$$\begin{aligned}
&e \in \mathrm{WF}(\omega + \kappa)\\
&\quad \leftrightarrow e \in P \;\&\; \neg(\exists p)(\delta_d(e, p) \notin \mathrm{WF}(\omega) \;\&\; (\exists x)\,(p <_e (x)_1\\
&\qquad <_e (x)_2 <_e \cdots <_e (x)_{k-1}))\\
&\quad \leftrightarrow e \in P \;\&\; (p)(\delta_d(e, p) \in \mathrm{WF}(\omega) \vee \neg(\exists x)(p <_e (x)_1 <_e \cdots\\
&\qquad <_e (x)_{k-1}).
\end{aligned}$$

Proposition 3.4. *For $n > 1$,*

(i) *$e \in \mathrm{WF}(\omega \cdot n)$ is Σ_{2n};*

(ii) *$e \in \mathrm{WF}(\omega \cdot n + k)$ is π_{2n+1} (as a predicate of e, k or fix k as a predicate of e).*

Proof. By induction on n.

$$e \in \mathrm{WF}(\omega \cdot (n+1)) \leftrightarrow (\exists k)\; e \in \mathrm{WF}(\omega \cdot n + k);$$

$$\begin{aligned}
e \in \mathrm{WF}(\omega \cdot (n+1) + k) \leftrightarrow e \in p \;\&\; \neg(\exists p)\,(\delta_d(e, p) \notin \mathrm{WF}(\omega \cdot (n+1)) \;\&\\
(\exists x)\,(p <_e (x)_1 <_e \cdots <_e (x)_{k-1})).
\end{aligned}$$

Let $e_n\{a_n\}$ be the m-degree of complete $\Sigma_n\{\pi_n\}$ sets. From Propositions 3.1, 3.3 and 3.4, we obtain

Theorem 3.5.

$$\begin{aligned}
&\deg_m \mathrm{WF}(k) \leqslant a_2 && \text{for} \quad k < \omega,\\
&\deg_m \mathrm{WF}(\omega \cdot n) \leqslant e_{2n} && \text{for} \quad \omega > n > 1,\\
&\deg_m \mathrm{WF}(\omega \cdot n + k) \leqslant a_{2n+1} && \text{for} \quad \omega > n \geqslant 1, \quad k < \omega.
\end{aligned}$$

[2] They correspond to δ, δ_1, δ_2 of [3] respectively.

4. The operations A, E, A_k and E_k

We shall define some operations on recursive partial orderings, which can not be restricted to recursive linear orderings because the results of the operations applied to recursive linear orderings in general are not linear.

Lemma 4.1. *Let $P_0, P_1, P_2, \ldots$ be an effective enumeration of some recursive partial orderings. Then there is a recursive partial ordering P_A such that P_A is well-founded iff (i) (P_i is well-founded); in that case* $\operatorname{rank}(P_A) = \sup_i(\operatorname{rank}(P_i))$.

Proof. Let $\langle x, y\rangle$ be a recursive 1–1 correspondence from $N \times N$ to N.
Define

$$\langle x_1, y_1\rangle P_A \langle x_2, y_2\rangle \quad \text{iff} \quad x_1 = x_2 \;\&\; y_1 P_{x_1} y_2.$$

Intuitively, we form P_A by putting P_i's mutually incomparably together.

Lemma 4.2. *$P_0, P_1, P_2, \ldots$ are as in Lemma 4.1. Then there is a recursive partial ordering P_E such that P_E is well-founded iff $(\exists i)$(P_i is well-founded); in that case* $\operatorname{rank}(P_E) \leq \operatorname{rank}(P_i) + i$ *for each i such that P_i is well-founded.*

Proof. Let $F(P)$ denote the field of the relation P. Define

$$\mathfrak{F}(P_E) = \{\langle a_0, a_1, \ldots, a_m\rangle \mid m \in N \;\&\; \forall i \leq m\; a_i \in \mathfrak{F}(P_i)\};^3$$

$$\begin{aligned}&\langle a_0, a_1, a_2, \ldots, a_m\rangle P_E \langle b_0, b_1, b_2, \ldots, b_n\rangle \quad \text{iff}\\ &\quad \langle a_0, a_1, a_2, \ldots, a_m\rangle, \langle b_0, b_1, b_2, \ldots, b_n\rangle \in \mathfrak{F}(P_E) \;\&\\ &\quad \{[m > n \;\&\; a_i <_{pi} bi\; \forall i \leq n] \quad \text{or}\\ &\quad \langle a_0, a_1, a_2, \ldots, a_m\rangle = \langle b_0, b_1, b_2, \ldots, b_n\rangle\},\end{aligned}$$

where $a <_p b$ denotes $aPb \;\&\; a \neq b$.

Clearly, P_E has a descending sequence iff each P_i has a descending sequence.

Suppose P_n is well-founded. By induction on $\operatorname{rank}(\langle a_0, a_1, \ldots, a_m\rangle)$ in P_E, we can show that $\operatorname{rank}(\langle a_0, a_1, \ldots, a_m\rangle)$ in P_E is less than or equal to $\operatorname{rank}(a_n)$ in P_n if $n \leq m$. Thus $\operatorname{rank}(\langle a_0, a_1, \ldots, a_m\rangle)$ in P_E is less than $\operatorname{rank}(P_n)$ if $m \geq n$. Now we shall show that,

$$\operatorname{rank}(\langle a_0, a_1, a_2, \ldots, a_{n-k}\rangle) \leq \operatorname{rank}(P_n) + k - 1, \quad \text{for} \quad k > 0,$$

by induction on k.

[3] $\langle a_0, a_1, \cdots, a_m\rangle$ denotes the sequence number corresponding to the sequence $a_0, a_1, \cdots, a_m$.

$$\text{rank}(\langle a_0, a_1, \ldots, a_{n-1}\rangle)$$
$$= \sup_{\langle \boldsymbol{b}\rangle <_{P_E} \langle \boldsymbol{a}\rangle} (\text{rank}(\langle b_1, \ldots, b_m\rangle) + 1)$$
$$\leqslant \sup_{b_n \in \mathfrak{F}(P_n)} (\text{rank}(b_n) + 1) = \text{rank}(P_n).$$
$$\text{rank}(\langle a_0, a_1, \ldots, a_{n-k}\rangle)$$
$$= \sup_{\langle \boldsymbol{b}\rangle <_{P_E} \langle \boldsymbol{a}\rangle} \{\text{rank}(\langle b_0, b_1, b_2, \ldots, b_{n-k+1}, \ldots, b_m\rangle) + 1\}$$
$$\leqslant \text{rank}(P_n) + (k-2) + 1$$

by induction hypothesis.

$$\text{rank}(P_E) = \sup_{\langle \boldsymbol{a}\rangle \in \mathfrak{F}(P_E)} (\text{rank}(\langle \boldsymbol{a}\rangle) + 1)$$
$$\leqslant \text{rank}(P_n) + (n-1) + 1$$
$$= \text{rank}(P_n) + n.$$

We define $A(P_0, P_1, P_2, \ldots) = P_A$ as in Lemma 4.1 and $E(P_0, P_1, P_2, \ldots) = P_E$ as in Lemma 4.2.

Now, suppose $P_1, P_2, \ldots, P_k$ are given recursive partial orderings. We define

$$A_k(P_1, P_2, \ldots, P_k) = A(\phi, P_1, P_2, \ldots, P_k, \phi, \phi, \ldots).$$

Finally, $E_k(P_1, P_2, \ldots, P_k)$ is defined by

$$\mathfrak{F}(E_k(P_1, P_2, \ldots, P_k)) = \{\langle a_1, \ldots, a_k\rangle \mid a_i \in \mathfrak{F}(P_i),\ i = 1, 2, \ldots, k\};$$
$$\langle a_1, \ldots, a_k\rangle\, E_k(P_1, P_2, \ldots, P_k) \times \langle b_1, \ldots, b_k\rangle \quad \text{iff}$$
$$[a_i <_{p_i} b_i \,\forall i = 1, 2, \ldots, k.] \quad \text{or} \quad \langle a_1, \ldots, a_k\rangle = \langle b_1, \ldots, b_k\rangle.$$

Then $E_k(P_1, P_2, \ldots, P_k)$ is well-founded iff at least one of the P_i's is well-founded and in that case

$$\text{rank}[E_k(P_1, P_2, \ldots, P_k)] = \min\{\text{rank}(P_i) \mid 1 \leqslant i \leqslant k \ \&\ P_i \text{ is well-founded}\}.$$

5. Strong reducibility

We say that A is strongly many-one reducible to $\text{WF}(\alpha)$, in notation $A \leqslant_m \text{WF}(\alpha)$ strongly, iff there exists a recursive function g such that

$$x \in A \to g(x) \in \text{WF}(\alpha),$$
$$x \notin A \to g(x) \in P - \text{WF}.$$

Thus, $A \leqslant_m \mathrm{WF}(\alpha)$ strongly implies A is uniformly many-one reducible[4] to $W(\sigma)$ for $\sigma \geqslant \alpha$.

Lemma 5.1. *If* $P(\boldsymbol{x}, y) \leqslant_m \mathrm{WF}(\alpha)$ *strongly, then*

$$(y)P(\boldsymbol{x}, y) \leqslant_m \mathrm{WF}((\alpha+1) \text{ strongly}.$$

Proof. Let f be a recursive function such that

$$P(\boldsymbol{x}, y) \rightarrow f(\boldsymbol{x}, y) \in \mathrm{WF}(\alpha),$$
$$\neg P(\boldsymbol{x}, y) \rightarrow f(\boldsymbol{x}, y) \in P-\mathrm{WF}.$$

Let $g(\boldsymbol{x})$ be a recursive function such that

$$\leqslant_{g(\boldsymbol{x})} = A(\leqslant_{f(\boldsymbol{x},0)}, \leqslant_{f(\boldsymbol{x},1)}, \ldots).$$

$$\begin{aligned}
(y)P(\boldsymbol{x}, y) &\rightarrow (y)(f(\boldsymbol{x}, y) \in \mathrm{WF}(\alpha)) \\
&\rightarrow (y)(\leqslant_{f(\boldsymbol{x},y)} \text{ is well-founded } \& \text{ rank } (\leqslant_{f(\boldsymbol{x},y)}) < \alpha) \\
&\rightarrow \leqslant_{g(\boldsymbol{x})} \text{ is well-founded } \& \text{ rank } (\leqslant_{g(\boldsymbol{x})}) \\
&\qquad = \sup_i \text{rank } (\leqslant_{f(\boldsymbol{x},i)})) \leqslant \alpha \\
&\rightarrow g(\boldsymbol{x}) \in \mathrm{WF}(\alpha+1), \\
\neg(y)P(\boldsymbol{x}, y) &\rightarrow (\exists y)(\leqslant_{f(\boldsymbol{x},y)} \text{ is not well-founded}) \\
&\rightarrow g(\boldsymbol{x}) \in P-\mathrm{WF}.
\end{aligned}$$

Lemma 5.2. *If* $P(\boldsymbol{x}, y) \leqslant_m \mathrm{WF}(\alpha)$ *strongly, then*

$$(\exists y)\, P(\boldsymbol{x}, y) \leqslant_m \mathrm{WF}(\alpha+\omega) \text{ strongly}.$$

Proof. Suppose $P(\boldsymbol{x}, y) \leqslant_m \mathrm{WF}(\alpha)$ strongly via f. Let

$$\leqslant_{g(\boldsymbol{x})} = E(\leqslant_{f(\boldsymbol{x},0)}, \leqslant_{f(\boldsymbol{x},1)}, \ldots)$$

where g is recursive. We have

$$\begin{aligned}
(\exists y)\, P(\boldsymbol{x}, y) &\rightarrow f(\boldsymbol{x}, y_0) \in \mathrm{WF}(\alpha) \quad \text{for some } y_0 \\
&\rightarrow \leqslant_{g(\boldsymbol{x})} \text{ is well-founded} \\
&\quad \& \text{ rank } (\leqslant_{g(\boldsymbol{x})}) \leqslant \text{rank } (\leqslant_{f(\boldsymbol{x},y_0)})) + y_0 \\
&\qquad < \alpha + \omega \\
&\rightarrow g(\boldsymbol{x}) \in \mathrm{WF}(\alpha+\omega), \\
\neg(\exists y)\, P(\boldsymbol{x}, y) &\rightarrow (y) \neg P(\boldsymbol{x}, y) \\
&\rightarrow (y) f(\boldsymbol{x}, y) \in P-\mathrm{WF} \\
&\rightarrow g(\boldsymbol{x}) \in P-\mathrm{WF}.
\end{aligned}$$

[4] For the definition of uniformly many-one reducible, cf. [3].

Theorem 5.3. *$n \geq 1$. R is recursive. Every predicate of the form*

$$(\exists x_n)(y_n)(\exists x_{n-1}) \cdots (\exists x_1)(y_1)\, R(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \boldsymbol{a})$$

is strongly many-one reducible to $\mathrm{WF}(\omega \cdot n)$. *Every predicate of the form*

$$(x_{n+1})(\exists x_n)(y_n) \cdots (\exists x_1)(y_1)\, R(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, \boldsymbol{a})$$

is strongly many-one reducible to $\mathrm{WF}(\omega \cdot n + 1)$.

Proof. Let $f(\boldsymbol{b})$ be a recursive function such that

$$\leq_{f(\boldsymbol{b})} = \begin{cases} \phi & \text{if} \quad R(\boldsymbol{b}), \\ S_{\geq} & \text{if} \quad \neg R(\boldsymbol{b}), \end{cases}$$

where $rS_{\geq}s$ iff $s \leq r$.

Then the recursive predicate R is strongly many-one reducible to WF(1) via f. Now the theorem immediately follows from Lemmas 5.1 and 5.2.

Corollary 5.4. $\omega > n \geq 1$,

$$e_{2n} \leq \deg_m \mathrm{WF}(\omega \cdot n),$$
$$a_{2n+1} \leq \deg_m \mathrm{WF}(\alpha)\, \forall \alpha \geq \omega \cdot n + 1.$$

We now study the m-degrees of $\mathrm{WF}(k)(k < \omega)$ & $\mathrm{WF}(\omega)$. The following Lemma is similar to Proposition 5.3 of [1].

Lemma 5.5. *Every predicate of the form $(x)(\exists y)R(x, y, a)$ where R is recursive is uniformly many-one reducible to $\mathrm{WF}(\sigma)$ for $\sigma \geq 1$.*

Proof. Define

$$\phi(x, y, a) = 1 \quad \text{if} \quad (\exists z)R(x, z, a),$$
$$\text{undefined} \quad \text{otherwise.}$$

By s-m-n then, there is a recursive function f such that

$$\{f(a)\}(x, y) = \phi(x, y, a).$$
$$(x)(\exists y)R(x, y, a) \to \{f(a)\}(x, y) \equiv 1$$
$$\to f(a) \in \mathrm{WF}(1),$$
$$\neg(x)(\exists y)R(x, y, a) \to \{f(a)\} \text{ is not total},$$
$$\to f(a) \notin \mathrm{WF}.$$

Thus, $a_2 \leq \deg_m \mathrm{WF}(\alpha)\, \forall \alpha > 0$. On the other hand $\mathrm{WF}(k)$ is π_2. So we obtain

Theorem 5.6. $\deg_m \mathrm{WF}(k) = a_2$, $0 < k < \omega$.

Let $a_{2,2}$ be the maximum m-degree of sets of the form $A \cap B$, where A is π_2 and B is Σ_2 (i.e., the degree $e_2 \wedge a_2$ of [1]). Then we have

Theorem 5.7. $\deg_m \mathrm{WF}(\omega) = a_{2,2}$.

Proof. $e \in \mathrm{WF}(\omega) \leftrightarrow e \in p$ and $(\exists k)(x)$

$$[\neg (x)_1 <_e (x)_2 <_e \cdots <_e (x)_k],$$

which is of the form $\pi_2 \wedge \Sigma_2$.

Now let A be any π_2 set and B be any Σ_2 set. By Lemma 5.5 and Theorem 5.3, *$A \leqslant_m \mathrm{WF}(\omega)$ via a recursive function f, $B \leqslant_m \mathrm{WF}(\omega)$ via a recursive function g. Define $\phi(u, x, y)$ with*

$$\begin{aligned} \phi(u, x, y) &\simeq \{f(u)\}(\pi_2(x), \pi_2(y)) && \text{if} \quad \pi_1(x) = \pi_1(y) = 1, \\ &\simeq \{g(u)\}(\pi_2(x), \pi_2(y)) && \text{if} \quad \pi_1(x) = \pi_1(y) = 2,^5 \\ &= 1 && \text{otherwise.} \end{aligned}$$

By s-m-n *theorem, there is a recursive function h such that*

$$\{h(u)\}(x, y) \simeq \phi(u, x, y).$$

$$\begin{aligned} u \in A \ \&\ u \in B &\to f(u) \in \mathrm{WF}(\omega) \ \&\ g(u) \in \mathrm{WF}(\omega) \\ &\to \leqslant_{h(u)} = A_2(\leqslant_{f(u)}, \leqslant_{g(u)}) \\ &\to \leqslant_{h(u)} = A_2(\leqslant_{f(u)}, \leqslant_{g(u)}) \\ &\to h(u) \in \mathrm{WF}(\omega), \\ \neg(u \in A \ \&\ u \in B) &\to u \notin A \vee u \notin B \\ &\to f(u) \notin \mathrm{WF}(\omega) \vee g(u) \notin \mathrm{WF}(\omega) \\ &\to h(u) \notin \mathrm{WF}(\omega). \end{aligned}$$

Thus, $A \cap B \leqslant_m \mathrm{WF}(\omega)$.

Combining the results of this section and the last section, we have for $1 < n < \omega$, $0 < k < \omega$

$$\begin{aligned} &\deg_m \mathrm{WF}(k) = a_2, \\ &\deg_m \mathrm{WF}(\omega) = a_{2,2}, \\ &\deg_m \mathrm{WF}(\omega \cdot n) = e_{2n}, \\ &\deg_m \mathrm{WF}(\omega \cdot n + k) = a_{2n+1}. \end{aligned}$$

The notion of strong reducibility is stronger than that of uniform reducibility. There exists some π_2 sets not strongly many-one reducible to WF(1), although all

[5] $\langle x, y\rangle$, $\pi_1(x)$, $\pi_2(x)$ are recursive functions such that $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$ and $\langle \pi_1(x), \pi_2(x)\rangle = x$.

π_2 sets are uniformly many-one reducible to $\mathrm{WF}(\sigma)$ for $\sigma \geq 1$. For if all π_2 sets were strongly many-one reducible to WF(1), then, by Lemma 5.2, $e_3 \leq \deg_m \mathrm{WF}(\omega) = a_{2,2}$, a contradiction.

6. Classification of WF(α) for $\alpha < \omega_1$ in hyperarithmetical hierarchy

We define the set of ordinal notations O and the sets $H(x)$ for $x \in O$ and hyperarithmetical hierarchy $\{\Sigma^0_\alpha\}_{\alpha<\omega_1}$ as in [6]. For $x \in O$, let $|x|$ denote the ordinal number for which x is a notation. Let $e_\alpha\{a_\alpha\}$ denote the many-one degree of complete Σ^0_α-sets $\{\pi^0_\alpha$-sets$\}$.

Define a recursive function f and a function r on recursive ordinals.

$$f(x) = \underbrace{2^{\cdot^{\cdot^{2}}}}_{2n} \quad \text{if} \quad x = \underbrace{2^{\cdot^{\cdot^{2}}}}_{n}, \quad n \geq 1,$$

$$= \underbrace{2^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}_{2n+1} \quad \text{if} \quad x = \underbrace{2^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}_{n}, \quad n \geq 0,$$

$$= 1 \quad \text{if} \quad x = 1,$$

$$= 0 \quad \text{otherwise.}$$

$$r(\omega \cdot \beta + n) = \omega \cdot \beta + 2n,$$

where $\beta < \omega_1$ and $n < \omega$.

We note that

$$x \in O \to f(x) \in O \;\&$$

$$\& \; |f(x)| = \begin{cases} r(|x|) & \text{if} \quad |x| \text{ is finite,} \\ r(|x|)+1 & \text{if} \quad |x| \text{ is infinite.} \end{cases}$$

Thus, $H(f(x))$ is a complete $\Sigma^0_{r((|x|)}$ set.

Theorem 6.1. *There is a partial recursive function $\phi(x, e)$ such that*

$$\operatorname{dom} \phi \supseteq O \times N$$

and

$$\forall x \in O - \{1, 2\} \, \forall e \; e \in \mathrm{WF}(\omega \cdot |x|) \leftrightarrow \phi(x, e) \in H(f(x)).$$

Thus, $\mathrm{WF}(\omega \cdot |x|) \leq_m H(f(x))$ *i.e.* $\mathrm{WF}(\omega \cdot |x|)$ *is* $\Sigma^0_{r(|x|)}$.

Proof. We shall define a partial recursive function $\phi(x, e)$ having the properties stated in the theorem with an index k_0, via the Recursion Theorem.

Define $\phi(2^2, e) = g(e)$ where $\mathrm{WF}(\omega \cdot 2) \leqslant_m \phi^{(4)}$ via g.

Suppose $x = 2^y$, $y \neq 0, 1, 2$. We have

$$\begin{aligned} e \in \mathrm{WF}(\omega \cdot |2^y|) &\leftrightarrow e \in \mathrm{WF}(\omega \cdot |y| + \omega) \\ &\leftrightarrow (\exists k) e \in \mathrm{WF}(\omega \cdot |y| + k) \\ &\leftrightarrow (\exists k)(p)\{\delta_d(e, p) \in \mathrm{WF}(\omega \cdot |y|) \\ &\qquad \vee (z)(\neg p <_e (z)_1 <_e \dots <_e (z)_{k-1})\} \\ &\leftrightarrow (\exists k)(p)\{\{k_0\}(y, \delta_d(e, p)) \in H(f(y)) \\ &\qquad \vee (z)(\neg p <_e (z)_1 <_e \dots <_e (z)_{k-1})\}, \end{aligned}$$

which is uniformly Σ_2 in $H(f(y))$ for $y \in O \backslash \{0, 1, 2\}$.

So

$$\mathrm{WF}(\omega \cdot |2^y|) \leqslant_m H''(f(y)) = H(f(2^y))$$

uniformly for $y \in O \backslash \{0, 1, 2\}$. Hence there is a partial recursive function $g_1(y, e)$ such that

$$\operatorname{dom} g_1 \supseteq O \times N$$

and

$$e \in \mathrm{WF}(\omega \cdot |2^y|) \leftrightarrow g_1(y, e) \in H(f(2^y)) \quad \forall y \in O \backslash \{0, 1, 2\}.$$

Define $\phi(2^y, e) \simeq g_1(y, e)$. Suppose $x = 3 \cdot 5^y$. Then

$$\begin{aligned} e \in \mathrm{WF}(\omega \cdot |3 \cdot 5^y|) &\leftrightarrow (\exists n)(e \in \mathrm{WF}(\omega \cdot |\phi_y(n)|)) \\ &\leftrightarrow (\exists n)(\{k_0\}(\phi_y(n), e) \in H(f(\phi_y(n)))) \\ &\leftrightarrow (\exists n)((\{k_0\}(\phi_y(n), e), f(\phi_y(n))) \in H(3 \cdot 5^y)), \end{aligned}$$

which is many-one reducible to $H(3 \cdot 5^y)$ uniformly in y, for $3 \cdot 5^y \in O$.

There is a partial recursive function $h(y, e)$ such that $3 \cdot 5^y \in O \rightarrow h(y, e)$ is defined and

$$e \in \mathrm{WF}(\omega \cdot |3 \cdot 5^y|) \leftrightarrow h(y, e) \in H(f(3 \cdot 5^y)).$$

Define $\phi(3 \cdot 5^y, e) \simeq h(y, e)$.

Remark. In the proof of Theorem 6.1, actually a partial recursive function $\phi(k, x, e)$ of three variables is defined; then we use the recursion theorem to pick up k_0 subject to $\{k_0\}(x, e) \simeq \phi(k_0, x, e)$ and finally we can prove $\operatorname{dom} \{k\} \supseteq O \times N$ by induction in O. We can get a similar result to Theorem 6.1 by using the Recursion Lemma (p. 398 of [6]). The result reads as follows: there is a partial recursive function Ψ subject to $(x)[x \in O \backslash \{1, 2\} \rightarrow \Psi(x)$ is defined and $\mathrm{WF}(\omega|x|) \leqslant_m H(f(x))$ via $\{\Psi(x)\}$.

7. Many-one degrees of WF(α)

We shall determine the many-one degrees of $\mathrm{WF}(\alpha)$ for all recursive ordinals α in this section.

Theorem 7.1. *There exists a recursive function* $g(x, n)$ *such that*

$$\forall x \in O - \{1, 2\}\ H(f(x)) \leqslant_m \mathrm{WF}(\omega \cdot |x|) \text{ strongly}$$

via $\lambda n g(x, n)$.

Proof. Again we use the Recursion Theorem to define such a $g(x, n)$ with an index k_0.

Case 1.

$$x = \underbrace{2^{\cdot^{\cdot^{2}}}}_{i}, \quad (i \geqslant 2).$$

Define

$$g(\underbrace{2^{\cdot^{\cdot^{2}}}}_{i}, n) = a(i, n),$$

where

$$H(\underbrace{2^{\cdot^{\cdot^{2}}}}_{2i}) \leqslant_n \mathrm{WF}(\omega \cdot i) \text{ strongly}$$

via $\lambda n a(i, n)$.

Case 2. $x = 3 \cdot 5^y$.

We shall first show that both $H(3 \cdot 5^y)$ and $\overline{H(3 \cdot 5^y)}$ are many-one reducible to $\mathrm{WF}(\omega \cdot |3 \cdot 5^y|)$ for $3 \cdot 5^y \in O$.

$$\begin{aligned} m \in H(3 \cdot 5^y) &\leftrightarrow \pi_2(m) <_o 3 \cdot 5^y \ \&\ \pi_1(m) \in H(\pi_2(m)) \\ &\leftrightarrow \pi_2(m) <_o \phi_y(a_0) \quad \text{for some } a_0 \\ &\qquad \&\ \pi_1(m) \in H(\pi_2(m)). \end{aligned}$$

Uniformly in y and a_0,

$$\pi_2(m) <_o \phi_y(a_0) \ \&\ \pi_1(m) \in H(\pi_2(m))$$

is r.e. in $H(\phi_y(a_0))$. So

$$\pi_2(m) <_o \phi_y(a_0) \ \&\ \pi_1(m) \in H(\pi_2(m))$$

is many-one reducible to $H(f(\phi_y(a_0)))$ via a recursive function $\lambda m g_1(y, a_0, m)$, where $g_1(y, a_0, m)$ is a partial recursive function with the property that

$3 \cdot 5^y \in O \to g_1(y, a_0, m)$ is defined. Now we suppose

$$H(f(\phi_y(a_0))) \leqslant_m \mathrm{WF}(\omega \cdot |\phi_y(a_0)|) \text{ strongly}$$

via $\lambda n\{k_0\}(\phi_y(a_0), n)$.

Define a recursive function $f_0(y, m)$ so that

$$\leqslant_{f_0(y,m)} = E(\leqslant_{\{k_0\}(\phi_y(a_0), g_1(y,a_0,m))} \mid a_0 = 0, 1, 2, \ldots),$$

whenever the right hand side is defined. Then,

$$H(3 \cdot 5^y) \leqslant_m \mathrm{WF}(\omega \cdot |3 \cdot 5^y|) \text{ strongly}$$

via $\lambda m f_0(y, m)$, for $3 \cdot 5^y \in O$.

$$m \notin H(3 \cdot 5^y) \leftrightarrow \neg \pi_2(m) <_0 3 \cdot 5^y \vee \neg \pi_1(m) \in H(\pi_2(m)).$$

Define a partial ordering $\leqslant_{y,m}$ as follows.

Generate elements which are $<_0 3 \cdot 5^y$, meanwhile generate natural linear ordering $\{0, 1, 2, \ldots\}$ as a part of $\leqslant_{y,m}$ in the following manner: once we generate a new element $<_0 3 \cdot 5^y$ and $\neq \pi_2(m)$, we put a natural number (in the natural order) into $\leqslant_{y,m}$. If the new element $<_0 3 \cdot 5^y$ generated is $\pi_2(m)$, stop generating the natural ordering (say $\{0, 1, 2, \ldots, t\}$ has been generated so far). Find an a_0 such that $\pi_2(m) <_0 \phi_y(a_0)$ by dovetailing. Since $\neg \pi_1(m) \in H(\pi_2(m))$ is recursive in $H(\phi_y(a_0))$, it is many-one reducible to $H(f(\phi_y(a_0)))$ via a function $\lambda m g_2(y, m)$, where g_2 is a partial recursive function which is defined when $3 \cdot 5^y \in O$. $H(f(\phi_y(a_0)))$ is supposed to be many-one reducible to $\mathrm{WF}(\omega \cdot |\phi_y(a_0)|)$ strongly via $\lambda n\{k_0\}(\phi_y(a_0), n))$. Thus

$$\neg \pi_1(m) \in H(\pi_2(m)) \leqslant_m \mathrm{WF}(\omega \cdot |\phi_y(a_0)|) \text{ strongly}$$

via $\lambda m\{k_0\}(\phi_y(a_0), g_2(y, m))$. Let $\leqslant_1$ be the partial ordering such that

$$x + t + 1 \leqslant_1 y + t + 1 \quad \text{iff} \quad \{\{k_0\}(\phi_y(a_0), g_2(y, m))\}(x, y) = 0.$$

Let $\leqslant_{y,m}$ be the ordering $\leqslant^{\uparrow}_{\{0,1,2,\ldots,t\}} \cup \leqslant_1$ where $\leqslant^{\uparrow}_{\{0,1,2,\ldots,t\}}$ is the natural ordering restricted to $\{0, 1, 2, \ldots, t\}$. Note if $\pi_2(m)$ never appears in the generating of elements $<_0 3 \cdot 5^y$, $\leqslant_{y,m}$ is the natural ordering $\leqslant^{\uparrow}$. Let $f_1(y, m)$ be a recursive function such that $\leqslant_{f_1(y,m)} = \leqslant_{y,m}$.

Then $\overline{H(3 \cdot 5^y)} \leqslant_m \mathrm{WF}(\omega \cdot |3 \cdot 5^y|)$ strongly via $\lambda m f_1(y, m)$. Now,

$$n \in H(f(3 \cdot 5^y)) \leftrightarrow n \in H(3 \cdot 5^y)' \leftrightarrow (\exists z)\, T^{H(3 \cdot 5^y)}(n, n, z).$$

For each z, consider $T^{H(3 \cdot 5^y)}(n, n, z)$. If n is not an oracle turning machine or z is not an oracle computation for some oracle set A of the machine n with input n, let $f(y, n, z)$ be a fixed Gödel number of $S_{\geqslant}$, the reverse of natural ordering (i.e., a descending sequence). Otherwise, let $m_1, m_2, \ldots, m_k$ be the numbers such that "$M_i \in A$?" asked in the computation z and $z_1, z_2, \ldots, z_k$ (either 0 or 1) the answers to the oracles according to the computation z.

Let $f(y, n, z)$ be a recursive function such that

$$\leqslant_{f(y,n,z)} = A_k(\leqslant_{f_{z1}(y,m_1)}, \leqslant_{f_{z2}(y,m_2)}, \ldots, \leqslant_{f_{zk}(y,m_k)}),$$

Thus,

$$\hat{z}T^{H(3\cdot 5^y)}(n, n, z) \leqslant_m \mathrm{WF}(\omega \cdot |3 \cdot 5^y|) \text{ strongly}$$

via $\lambda z f(y, n, z)$. Let $h(y, n)$ be a recursive function such that

$$\leqslant_{h(y,n)} = E(\leqslant_{f(y,n,z)} | \, z = 0, 1, 2, \ldots)$$

whenever the right hand side is defined. Define $g(3 \cdot 5^y, n) = h(y, n)$.

Case 3.

$$x = \underbrace{2^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}_{n}, \quad n \geqslant 1.$$

$$e \in H(f(x)) \leftrightarrow e \in H^{(2n+1)}(3 \cdot 5^y) \leftrightarrow$$
$$(\exists x_n)(y_n)(\exists x_{n-1})(y_{n-1}) \cdots (\exists x_1)(y_1)(\exists x_0)$$
$$R^{H(3\cdot 5^y)}(e, x_0, x_1, y_1, \ldots, x_n, y_n),$$

where $R^{H(3\cdot 5^y)}$ is a predicate recursive in $H(3 \cdot 5^y)$.

Now, $\exists x_0 \, R^{H(3\cdot 5^y)}(e, x_0, x_1, y_1, \ldots, x_n, y_n)$ is many-one reducible to $H(3 \cdot 5^y)'$ via a recursive function

$$\lambda x_1, y_1, \ldots, x_n, y_n, e \, h(y, x_1, y_1, \ldots, x_n, y_n, e),$$

where h is a partial recursive function which is defined (if $3 \cdot 5^y \in O$, and $H(3 \cdot 5^y)'$ is supposed strongly many-one reducible to $\mathrm{WF}(\omega \cdot |3 \cdot 5^y|)$ via $\lambda n\{k_0\}(3 \cdot 5^y, n)$. Let $h^*(n, y, e)$ be a recursive function such that

$$\leqslant_{h^*(n,y,e)} = E_{x_n} A_{y_n} \cdots E_{x_1} A_{y_1}(\{\leqslant_{\{k_0\}(3\cdot 5^y, h(y, x_1, y_1, \ldots, x_n, y_n, e))}\}),$$

if the right-hand side is defined.

Define $g(x, e) = h^*(n, y, e)$.

$$e \in H(f(x)) \leqslant_m \mathrm{WF}(\omega \cdot (|3 \cdot 5^y| + n)) = \mathrm{WF}(\omega \cdot |x|) \text{ strongly}$$

via $\lambda e g(x, e)$.

Case 4. Otherwise. Define $g(x, e) = 0$. Combining Theorems 6.1 and 7.1, we obtain

$$\deg_m \mathrm{WF}(\omega \cdot |x|) = \deg_m H(f(x)) = e_{r(|x|)} \quad \forall x \in O - \{1, 2\},$$

i.e., $\deg_m \mathrm{WF}(\omega \cdot \alpha) = e_{r(\alpha)}$ for every recursive, ordinal $\alpha > 1$.

Theorem 7.2.

$$\mathrm{WF}(\omega \cdot |x| + k) \equiv_m \overline{H(2^{f(x)})} \quad \forall x \in 0 - \{1\}, \quad 0 < k < \omega.$$

Proof. Let

$$x = \underbrace{2^{\cdot^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}}_{n}$$

$$\begin{aligned} e \in \mathrm{WF}(\omega \cdot |x| + k) \leftrightarrow\ & e \in P\ \&\ \neg(\exists p) \\ & \times(\delta_d(e, p) \notin (\mathrm{WF}(\omega \cdot |x|)\ \&\ (\exists z) \\ & \times(p <_e (z)_1 <_e \cdots <_e (z)_{k-1})) \\ \leftrightarrow\ & e \in P\ \&\ (p)(\delta_d(e, p) \in \mathrm{WF}(\omega \cdot |x|) \vee \\ & \neg(\exists z)(p <_e (z)_1 <_e \cdots <_e (z)_{k-1})), \end{aligned}$$

which is π_1 in $H(f(x))$, thus $\leqslant_m \overline{H(f(x))'}$. For the other direction, $e \in H(2^{f(x)})$ can be written in the form

$$(\exists x_{n+1})(y_{n+1})(\exists x_n)(y_n) \cdots (\exists x_1)(y_1) R^{H(3\cdot 5^y)},$$

where $R^{H(3\cdot 5^y)}$ is a predicate recursive in $H(3 \cdot 5^y)$. So

$$\neg e \in H(2^{f(x)}) \leftrightarrow (x_{n+1})(\exists y_{n+1})(x_n)(\exists y_n) \cdots (x_1)(\exists y_1) \neg R^{H(3\cdot 5^y)}.$$

$(\exists y_1)\neg R^{H(3\cdot 5^y)}$ is r.e. in $H(3 \cdot 5^y)$. So it is strongly many-one reducible to $\mathrm{WF}(\omega \cdot |3 \cdot 5^y|)$ by Theorem 7.1. Repeatly using Lemmas 5.1, 5.2, we obtain

$$\overline{H(2^{f(x)})} \leqslant_m \mathrm{WF}(\omega \cdot |3 \cdot 5^y| + \omega \cdot n + 1) \text{ strongly.}$$

As a direct consequence of Theorem 7.2, we have

$$\deg_m \mathrm{WF}(\omega \cdot \alpha + k) = a_{r(\alpha)+1}$$

for every recursive ordinal $\alpha > 0$. Thus, we have determined the many-one degrees of all $\mathrm{WF}(\alpha)$ as stated in the Table 1 of Section 1.

8. Classification of $W(\omega^\alpha)$ in hyperarithmetical hierarchy

We now aim to determine the many-one degrees of all $W(\alpha)$. In [3] it was proved that $W(\omega^n)$ is Σ_{2n} for all finite n. In this section, we will extend the result to all recursive ordinals α.

Theorem 8.1. *There is a partial recursive function $\phi(x, e)$ such that* dom $\phi \supseteq O \times N$ *and*

$$\forall x \in O - \{1, 2\}\ \forall e\ [e \in W(\omega^{|x|}) \leftrightarrow \phi(x, e) \in H(f(x))].$$

Hence

$$W(\omega^{|x|}) \leqslant_m H(f(x)) \quad \forall x \in O - \{1, 2\}.$$

Proof. Once more we use Recursion Theorem to define the function $\phi(x, e)$ with an index k_0.

Define $\phi(2^2, e) = g(e)$ where $g(e)$ is a recursive function such that

$$e \in W(\omega^2) \leftrightarrow g(e) \in \phi^{(4)} = H(f(2^2)).$$

Suppose $x = 2^y$, $y \neq 0, 1, 2$.

$$\begin{aligned} e \in W(\omega^{|x|}) &\leftrightarrow e \in W(\omega^{|y|} \cdot \omega) \\ &\leftrightarrow e \in W(\omega^{|y|} + 1) \vee \\ &\quad \{e \in L \ \& \ (\exists z)\{\delta_d(e, (z)_0) \in W(\omega^{|y|} + 1) \ \& \\ &\quad \delta_u(e, (z)_{1h(z) \dot- 1}) \in W(\omega^{|y|} + 1) \ \& \\ &\quad (u)_{<1h(z) \dot- 1}[(z)_u <_e (z)_{u+1} \ \& \\ &\quad \delta_b(e, (z)_u, (z)_{u+1}) \in W(\omega^{|y|} + 1)]\}\}, \end{aligned}$$

where

$$\begin{aligned} e \in W(\omega^{|y|} + 1) &\leftrightarrow e \in L \ \& \ (z)(z \leqslant_e z \rightarrow \delta_d(e, z) \in W(\omega^{|y|})) \\ &\leftrightarrow e \in L \ \& \ (z)(z \leqslant_e z \rightarrow \{k_0\}(y, \delta_d(e, z)) \in H(f(y))). \end{aligned}$$

So $e \in W(\omega^{|x|}) \leqslant_m H(f(y))''$ uniformly in y for $y \in O$. We can construct a partial recursive function $g(y, e)$ with domain $\supseteq O \times N$ such that

$$e \in W(\omega^{|x|}) \leftrightarrow g(y, e) \in H(f(y))'' = H(f(x)).$$

Define $\phi(2^y, e) \simeq g(y, e)$.

Suppose $x = 3 \cdot 5^y$

$$\begin{aligned} e \in W(\omega^{|x|}) &\leftrightarrow (\exists n)(e \in W(\omega^{|\phi_y(n)|})) \\ &\leftrightarrow (\exists n)(\{k_0\}(\phi_y(n), e) \in H(f(\phi_y(n)))) \\ &\leftrightarrow (\exists n)(\langle\{k_0\}(\phi_y(n), e), f(\phi_y(n))\rangle \in H(x)), \end{aligned}$$

which is uniformly r.e. in $H(x)$ for $X = 3 \cdot 5^y \in O$. So there is a partial recursive function $h(y, e)$ such that if $3 \cdot 5^y \in O$, then $h(y, e)$ is defined and

$$e \in W(\omega^{|x|}) \leftrightarrow h(y, e) \in H(x)' = H(f(x)).$$

Define $\phi(3 \cdot 5^y, e) \simeq h(y, e)$.

It immediately follows that $W(\omega^\alpha)$ is $\Sigma^0_{r(\alpha)}$ for $\alpha > 1$.

9. A reduction of partial orderings to linear orderings

From the results at the end of Section 5 and those in [3] and [1], we see that $\mathrm{WF}(\omega \cdot n) \equiv_m W(\omega^n)$ for $0 < n < \omega$. It is natural to conjecture that $\mathrm{WF}(\omega \cdot \alpha) \equiv_m W(\omega^\alpha)$ for every recursive ordinal α and to expect that there is an effective procedure to construct a recursive linear ordering L for each recursive partial ordering P such that L is well-ordered iff P is well-founded and in that case if rank $(p)<\omega\cdot\alpha$, then rank $(L)<\omega^\alpha$. After a certain amount of investigation, the existence of such effective procedures (which can establish the strong reducibility of $H(f(x))$ to $W(\omega^{|x|})$) is in doubt, although the conjecture $WF(\omega\cdot\alpha)\equiv_m W(\omega^\alpha)$ for $0<\alpha<\omega_1$ is true as we shall see in Section 10. The following Lemma is the best result that has been established so far in concerning such kind of effective procedures.

Lemma 9.1. *For every recursive partial ordering P, we can effectively construct a recursive linear ordering L uniformly in P such that P is well-founded iff L is well-ordered and in that case*

if rank $(P)=0$, *then* $|L|=0$,
if rank $(P)=\omega\cdot(\omega\cdot\beta+k)+i>0$, k, i *finite, then*

$$\omega^{\omega\cdot\beta+k}\leqslant|L|\leqslant\begin{cases}\omega^{(\omega\cdot\beta+2k)} & \text{if } i=0,\\ \omega^{(\omega\cdot\beta+2k)}\cdot\omega\cdot 2^i & \text{if } i>0.\end{cases}$$

Proof. Define an r.e. set A_p and a recursive linear ordering R on the set of all sequence numbers

$$A_p=\{\langle a_0,a_1,a_2,\ldots,a_n\rangle\mid n\geqslant 0,(\mathrm{i})_{\leqslant n}(a_i\in\mathfrak{F}(P))\ \&\ (\mathrm{i})_{\leqslant n-1}(\exists x)[a_{i+1}P_<(x)_1P_<(x)_2P_<\cdots P_<(x)_{a_i}P_<a_i]\},$$

where $yP_<z$ iff yPz & $y\neq z$.

$$\langle a_0,a_1,a_2,\ldots,a_n\rangle R\langle b_0,b_1,b_2,\ldots,b_m\rangle \text{ iff } (\exists t)\leqslant\min(n,m)[(s)_{<t}(a_s=b_s)\ \&\ a_t<b_t]\ [n\geqslant m\ \&\ (t)_{\leqslant m}(a_t=b_t)].$$

We claim that P is well-founded if $R\upharpoonright_{A_p}$ is well-ordered. Suppose P has a descending sequence $c_0,c_1,c_2,\ldots$. Define $k_0=0$, $k_{n+1}=k_n+C_{k_n}+1$. Let

$$S_n=\langle C_{K_0},C_{k_1},\ldots,C_{k_n}\rangle\in A_p.$$

S_0, S_1, S_2, ... is a descending sequence in $R\upharpoonright_{A_p}$. Conversely, let $\{S_k\}_{k=0,1,2,\ldots}$ be a descending sequence in $R\upharpoonright_{A_p}$ where $S_k=\langle a_0^k,a_1^k,\ldots,a_{ik}^k\rangle$. Since for every k, $a_0^k\leqslant a_0^0$, there are infinitely many k such that $a_0^k=b_0$ where b_0 is a fixed number less than or equal to a_0^0. Let $C=\{S_0,S_1,S_2,\ldots\}$. Define $C_0=\{S_k\in C\mid a_0^k=b_0\}$. Suppose b_n, c_n have been defined such that C_n is infinite and $(S_k)\in C_n$ $(\mathrm{i})\leqslant n$ $(a_i^k=b_i)$. Let S_{jn} be the first element in C_n of length greater than or equal to $n+2$. For every S_i in C_n that $S_iP_<S_{jn}$, we have $a_{n+1}^i\leqslant a_{n+1}^{jn}$.

So there is $b_{n+1}\leqslant a_{n+1}^{jn}$ with the property that there are infinitely many elements S_k in C_n such that $a_{n+1}^k=b_{n+1}$. Let $C_{n+1}=\{S_k\in C_n\mid a_{n+1}^k=b_{n+1}\}$. Then

$b_0, b_1, \ldots$ is a descending sequence in P. We now show that

$$\operatorname{rank}(P) = \omega \cdot (\omega \cdot \beta + k) + i \rightarrow$$
$$|R \upharpoonright_{A_p}| \leqslant \begin{cases} \omega^{(\omega \cdot \beta + 2k)} & \text{if } i = 0, \\ \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^i & \text{if } i > 0, \end{cases}$$

by induction on rank (P).

Base. rank $(P) = 0$. Trivial.

Induction step. rank $(P) > 0$. Let

$$B_j = \{\langle a_0, a_1, \ldots, a_n \rangle \in A_p \mid a_0 = j\},$$
$$R \upharpoonright_{A_p} = R \upharpoonright_{B_0}, R \upharpoonright_{B_2}, \ldots.$$

Let

$$P_j = \{a \in \mathfrak{F}(P) \mid (\exists x) a P_<(x)_1 P_<(x)_2 P_< \cdots P_<(x)_j P_< j\}.$$

Define $\beta \dot{-} j = \omega \cdot \alpha + (n \dot{-} j)$ if $\beta = \omega \cdot \alpha + n$. We have rank $(P_j) \leqslant \operatorname{rank}(j) \dot{-} j <$ rank (p). Form A_{p_j} from P_j as A_p from P. Then

$$B_j = \{\langle j \rangle\} \cup \{\langle j, a_0, a_1, \ldots, a_n \rangle \mid \langle a_0, a_1, \ldots, a_n \rangle \in A_{p_j}\} \quad \text{if } j \in \mathfrak{F}(P),$$
$$= \phi \quad \text{if } j \notin \mathfrak{F}(P).$$

$$\operatorname{rank}(R \upharpoonright_{B_j}) = \operatorname{rank}(R \upharpoonright_{A_{p_j}}) + 1 \quad \text{if } j \in \mathfrak{F}(P),$$
$$= 0 \quad \text{if } j \notin \mathfrak{F}(P).$$

If $i > 0$,

$$\operatorname{rank}(P_j) \leqslant \operatorname{rank}(j) \dot{-} j$$
$$\leqslant \begin{cases} \omega \cdot (\omega \cdot \beta + k) + (i - 1 - j) & \text{if } j \leqslant i - 1, \\ \omega \cdot (\omega \cdot \beta + k) & \text{if } j > i - 1. \end{cases}$$

$$\begin{aligned} \operatorname{rank}(R \upharpoonright_{A_p}) &= \sum_{j<\omega} \operatorname{rank}(R \upharpoonright_{B_j}) \\ &\leqslant \sum_{j<\omega} (\operatorname{rank}(R \upharpoonright_{A_{p_j}}) + 1) \\ &\leqslant \sum_{j=0}^{i-1} (\omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^{(i-1)-j} + 1) \\ &\quad + \sum_{i \leqslant j < \omega} (\omega^{(\omega \cdot \beta + 2k)} + 1) \\ &= \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^{i-1} + \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^{i-2} + \ldots \\ &\quad + \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^1 + \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^0 + \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \\ &= \omega^{(\omega \cdot \beta + 2k)} \cdot \omega \cdot 2^i. \end{aligned}$$

If $i=0$,

$$\operatorname{rank}(P_j) \leqslant \operatorname{rank}(j) \dot{-} j$$

$$\leqslant \begin{cases} \omega \cdot [\omega \cdot \beta + (k-1)] + m_j & \text{for some } m_j < \omega \quad \text{if} \quad k > 0, \\ \omega \cdot (\omega \cdot r_j + k_j) & \text{for some } r_j < \beta,\, k_j < \omega \quad \text{if} \quad k = 0. \end{cases}$$

$$\operatorname{rank}(R \upharpoonright_{A_p}) = \sum_{j<\omega} \operatorname{rank}(R \upharpoonright_{B_j}) \leqslant \sum_{j<\omega} [\operatorname{rank}(R \upharpoonright_{A_{pj}}) + 1]$$

$$\leqslant \begin{cases} \sum_{j<\omega} [\omega^{\omega \cdot \beta + 2(k-1)} \cdot \omega \cdot 2^{m_j} + 1] = \omega^{\omega \cdot \beta + 2k} & \text{if} \quad k > 0, \\ \sum_{j<\omega} [\omega^{\omega r_j + 2k_j} + 1] \leqslant \omega^{\omega \cdot \beta} & \text{if} \quad k = 0. \end{cases}$$

Next we show $\operatorname{rank}(R \upharpoonright_{A_p}) \geqslant \omega^{\omega \cdot \beta + k}$ by induction on $\operatorname{rank}(P)$.

Base. $\operatorname{rank}(P)$ is finite. $|R \upharpoonright_{A_p}| \geqslant 1$.

Induction step (1). $\operatorname{rank}(P) = \omega \cdot (\omega \cdot \beta + k)$, $k > 0$. There exist infinitely many elements $i_1, i_2, \ldots, i_n, \ldots$ such that $\operatorname{rank}(i_n) \geqslant \omega \cdot (\omega \cdot \beta + k - 1)$.

$$\operatorname{rank}(R \upharpoonright_{A_p}) \geqslant \sum_{n<\omega} \operatorname{rank}(R \upharpoonright_{B_{i_n}})$$

$$\geqslant \sum_{n<\omega} \operatorname{rank}(R \upharpoonright_{A_{p_{i_n}}})$$

(where $\operatorname{rank}(P_{i_n}) \geqslant \omega \cdot (\omega \cdot \beta + k - 1)$)

$$\geqslant \sum_{n<\omega^\omega} \omega \cdot \beta + k - 1$$

$$= \omega^{\omega \cdot \beta + k}.$$

(2). $\operatorname{rank}(P) = \omega \cdot (\omega \cdot \beta)$. There is a sequence of ordinals $\gamma_1, \gamma_2, \ldots$ such that $\lim_j \gamma_j = \omega \cdot \beta$ and there is a sequence of elements $i_i, i_2, \ldots$ in $\mathfrak{F}(P)$ such that $\operatorname{rank}(i_n) \geqslant \omega \cdot \gamma_n$.

$$\operatorname{rank}(R \upharpoonright_{A_p}) \geqslant \operatorname{rank}(R \upharpoonright_{B_{i_n}}) \geqslant \operatorname{rank}(R \upharpoonright_{A_{p_{i_n}}}) \geqslant \omega^{\gamma_n} \quad \forall n.$$

So $\operatorname{rank}(R \upharpoonright_{A_p}) \geqslant \omega^{\omega \cdot \beta}$.

(3). $\operatorname{rank}(P) = \omega \cdot (\omega \cdot \beta + k) + i$, $i > 0$. There is i_0 such that $\operatorname{rank} i_0 = \omega \cdot (\omega \cdot \beta + k)$.

$$\operatorname{rank}(R \upharpoonright_{A_p}) \geqslant \operatorname{rank}(R \upharpoonright_{B_{i_0}}) \geqslant \operatorname{rank}(R \upharpoonright_{A_{p_{i_0}}})$$

(where $\operatorname{rank}(P_{i_0}) = \omega \cdot (\omega \cdot \beta + k)$)

$$\geqslant \omega^{\omega \cdot \beta} + k.$$

$R \upharpoonright_{A_p}$ is an r.e. linear ordering. By [8], we can construct a recursive linear ordering L uniformly in $R \upharpoonright_{A_p}$, hence uniformly in P, such that $L \cong R \upharpoonright_{A_p}$. We note that $\operatorname{rank}(P) = \omega \cdot (\omega \cdot \beta) \neq 0$ implies $|L| = \omega^{\omega \cdot \beta}$.

Let's denote the L in Lemma 9.1 by $T(P)$. Then there is a recursive function $t(e)$ such that $\leqslant_{t(e)} = T(\leqslant_e)$ if $e \in P$. Similarly to the case for $\mathrm{WF}(\alpha)$, we define $A \leqslant_m W(\alpha)$ strongly iff there exists a recursive function g such that $x \in A \to g(x) \in W(\alpha)$, $x \notin A \to g(x) \in L - W$. Then we have

Corollary 9.2. *If $|x|$ $(x \in O)$ is a limit ordinal, then $H(f(x)) \leqslant_m W(\omega^{|x|})$ strongly.*

Proof. By Theorem 7.1, $H(f(x)) \leqslant_m \mathrm{WF}(\omega \cdot |x|)$ strongly via a recursive function g. Lemma 9.1 gives

$$e \in \mathrm{WF}(\omega \cdot |x|) \to t(e) \in W(\omega^{|x|}),$$
$$e \in P - \mathrm{WF} \to t(e) \in L - W.$$

Thus, $H(f(x)) \leqslant_m W(\omega^{|x|})$ strongly via $t \circ g$.
By Corollary 9.2 and Theorem 8.1, $H(f(x)) \equiv_m W(\omega^{|x|})$ if $|x|$ is a limit. Hence, $\deg_m W(\omega^{\alpha}) = e_{r(\alpha)}$ if α is limit.

Corollary 9.3. *$H(f(x)) \leqslant_m W(\omega^{|f(x)| \dot{-} 1})$ strongly for $|x| > 1$.*

Proof. Use Theorem 7.1 and the facts

$$e \in \mathrm{WF}(\omega \cdot |x|) \to t(e) \in W(\omega^{|f(x)| \dot{-} 1}),$$
$$e \in P - \mathrm{WF} \to t(e) \in L - W.$$

If $|x| = \omega \cdot \beta + k$, then $|f(x)| \dot{-} 1 = |x| + k$, which is larger than what we want.
If $|x|$ is an infinite successor recursive ordinal, "$H(f(x)) \leqslant_m W(\omega^{|x|})$ strongly?" is still an open question.

10. Nice reducibility

In order to establish $H(f(x)) \leqslant_m W(\omega^{|x|})\ \forall x \in O - \{1, 2\}$, we need a new type of m-reducibility. We define $A \leqslant_m \mathrm{WF}(\alpha)$ (resp. $W(\alpha)$) *nicely* iff there is a recursive function g such that

$$x \in A \to g(x) \in \mathrm{WF}(\alpha) \text{ (resp. } W(\alpha)),$$
$$x \in A \to g(x) \in \mathrm{WF} \ \&\ \mathrm{rank}\,(g(x)) = \alpha$$
$$\text{(resp. } g(x) \in W \ \&\ |g(x)| = \alpha).$$

Theorem 10.1. *$H(f(x)) \leqslant_m \mathrm{WF}(\omega \cdot |x|)$ nicely $\forall x \in O - \{1, 2\}$.*

Proof. Given $x \in O - \{1, 2\}$, we can construct an r.e. well-ordered set R_x with rank $\omega \cdot |x|$:

$$\langle a, b\rangle R_x \langle c, d\rangle \quad \text{iff} \quad [a <_o c <_o x \quad \text{or} \quad (a = c <_o x \ \&\ b \leqslant d)].$$

By [8], we can construct a recursive well-ordering S_x such that $S_x \simeq R_x$.

Let $h(x,n)$ be a recursive function such that $\leqslant_{h(x,n)} = E_2(\leqslant_{g(x,n)}, S_x)$, where $g(x,n)$ is the recursive function in Theorem 7.1. Then we have

$$\begin{aligned} n \in H(f(x)) &\to g(x,n) \in \mathrm{WF}(\omega \cdot |x|) \\ &\to h(x,n) \in \mathrm{WF}(\omega \cdot |x|), \\ n \notin H(f(x)) &\to g(x,n) \in P - \mathrm{WF} \\ &\to \operatorname{rank}(h(x,n)) = \operatorname{rank}(S_x) = \omega \cdot |x|. \end{aligned}$$

So, $H(f(x)) \leqslant_m \mathrm{WF}(\omega \cdot |x|)$ nicely via $\lambda n h(x,n)$.

Theorem 10.2. *$H(f(x)) \leqslant_m W(\omega^{|x|})$ nicely $\forall x \in O - \{1, 2\}$.*

Proof. Case 1. X is finite. By the proof of Theorem 3(a) of [3].

Case 2. x is a limit. We note that

$$\begin{aligned} &e \in \mathrm{WF}(\omega \cdot |x|) \to t(e) \in W(\omega^{|x|}), \\ &\operatorname{rank}(e) = \omega \cdot |x| \to \operatorname{rank}[t(e)] = \omega^{|x|}. \end{aligned}$$

So $H(f(x)) \leqslant_m W(\omega^{|x|})$ nicely via $\lambda n t(h(x,n))$ where h is the recursive function defined in the proof of Theorem 10.1.

Case 3.

$$x = \underbrace{2^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}_{n} \qquad n > 0.$$

$a \in H(f(x))$ is of the form

$$(\exists x_1)(y_1)(\exists x_2)(y_2) \cdots (\exists x_n)(y_n)(\exists x_{n+1}) R^{H(3\cdot 5^y)}(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, a),$$

which is equivalent to

$$\begin{aligned} &\neg(x_1)(\exists x_1)(x_2)(\exists y_2) \cdots (x_n)(\exists y_n)(x_{n+1}) \\ &\quad \neg R^{H(3\cdot 5^y)}(x_1, y_1, \ldots, x_n, y_n, x_{n+1}, a). \end{aligned}$$

Now, by Lemma 2 of [3], it is equivalent to

$$\neg(Uz_1)(Uz_2) \cdots (Uz_n)(z_{n+1}) S^{H(3\cdot 5^y)}(z_1, z_2, \ldots, z_n, z_{n+1}, a)$$

for some $S^{H(3\cdot 5^y)}(z_1, z_2, \ldots, z_n, z_{n+1}, a)$ recursive in $H(3 \cdot 5^y)$.

$$\neg(z_{n+1}) S^{H(3\cdot 5^y)}(z_1, z_2, \ldots, z_{n+1}, a)$$

is r.e. in $H(3 \cdot 5^y)$. So

$$\begin{aligned} \neg(z_{n+1}) S^{H(3\cdot 5^y)}(z_1, z_2, \ldots, z_{n+1}, a)&[\leqslant_m H'(3 \cdot 5^y)] \\ &\leqslant_m W(\omega^{|3\cdot 5^y|}) \text{ nicely} \end{aligned}$$

via a recursive function g.

Define a well-ordering R_a on the set of $(n+1)$-tuples $\langle z_1, z_2, \ldots, z_n, z\rangle$ such that $z \in \mathfrak{F}(\leqslant g(z_1, z_2, \ldots, z_n, a))$ by

$$\langle z_1, z_2, \ldots, z_n, z\rangle R_a \langle v_1, v_2, \ldots, v_n, v\rangle$$
$$\text{iff}\quad [(\exists i_0)_1 \leqslant i_0 \leqslant n((t)_{<i_0}(z_t = v_t) \;\&\; z_{i_0} < v_{i_0})] \vee (z_i = v_i, \forall i = 1, 2, \ldots n) \;\&$$
$$z \leqslant_{g(z_1, z_2, \ldots, z_n, a)} v].$$

Let $h_1(a)$ be a recursive function such that $\leqslant_{h_1(a)} = R_a$.

$$(Uz_1)(Uz_2)\cdots(Uz_n)(z_{n+1})S^{H(3\cdot 5^y)}(z_1, \ldots, z_{n+1}, a)$$
$$\rightarrow |R_a| = \omega^{(|3\cdot 5^y|+n)},$$
$$\neg(Uz_1)(Uz_2)\cdots(Uz_n)(z_{n+1})S^{H(3\cdot 5^y)}(z_1, \ldots, z_{n+1}, a)$$
$$\rightarrow |R_a| < \omega^{(|3\cdot 5^y|+n)}.$$

Thus,

$$a \in H(f(x)) \rightarrow |R_a| < \omega^{|3\cdot 5^y|+n}$$
$$\rightarrow h_1(a) \in W(\omega^{|3\cdot 5^y|+n}) = W(\omega^{|x|}),$$
$$a \notin H(f(x)) \rightarrow |h_1(a)| = |R_a| = \omega^{|3\cdot 5^y|+n} = \omega^{|x|}.$$

Hence $H(f(x)) \leqslant_m W(\omega^{|x|})$ nicely via h_1.

11. Degrees of $W(\alpha)$

Combinating Theorems 8.1 and 10.2, we have $\deg_m W(\omega^{|x|}) = \deg_m H(f(x))$ $\forall x \in O - \{1, 2\}$ i.e. $\deg_m W(\omega^\alpha) = e_{r(\alpha)}$. By comparing their m-degrees, $W(\omega^\alpha) \equiv_m$ $\mathrm{WF}(\omega \cdot \alpha)$ for $0 < \alpha < \omega_1$.

We now extend the notion of limit points of order n in [3] to transfinite orders by inductive definition in O of $L_x(z, e)$ which asserts z is a limit point of $\leqslant_e$ of order $|x|$.

For $x = 1$:

$$L_1(z, e) \equiv z \leqslant_e z,$$

for $x = 2^y$ $(y \in O)$:

$$L_x(z, e) \equiv (\exists u)(u <_e z) \;\&\; (u)(u <_e z$$
$$\rightarrow (\exists v)(L_y(v, e) \;\&\; u <_e v <_e z)),$$

for $x = 3 \cdot 5^y \in O$:

$$L_x(z, e) \equiv (\exists u)(u <_e z) \;\&\; (u)(u <_e z$$
$$\rightarrow (k)(\exists v)(L_{\Phi_y(k)}(v, e) \;\&\; u <_e v <_e z)).$$

We first show that $L_x(z, e)$ is many-one reducible to $\overline{H(f(x))} \forall x \in O - \{1\}$.

Lemma 11.1. *There is a partial recursive function* $\phi(x, z, e)$ *such that* $\operatorname{dom}\phi \geqslant O \times N^2$ and $(z, e) \in L_x$ *iff* $\phi(x, z, e) \in \overline{H(f(x))} \forall x \in O - \{1\}$.

Proof. Define a partial recursive function ϕ with an index k_0 via the Recursion Theorem.

Case 1. $x = 2$.

$$L_2(z, e) \equiv (\exists u)(u <_e z) \;\&$$
$$\& \; (u)(u <_e z \to (\exists v)(v \leqslant_e v \;\&\; u <_e v <_e z)),$$

which is π_2. So $L_2 \leqslant_m \overline{\phi''} = \overline{H(f(2))}$ via a recursive function $g(z, e)$. Define $\phi(2, z, e) = g(z, e)$.

Case 2. $x = 2^y$. Since $L_x(z, e)$ is π_2 in L_y, $\neg L_x$ is Σ_2 in L_y (uniformly in $y \in O$). So $\neg L_x \leqslant_m Ly''$ via a partial recursive function $g_1(y, z, e)$ with domain $\supseteq O \times N^2$. It is supposed that $L_y \leqslant_m \overline{H(f(y))}$ via $\{k_0\}(y, z, e)$. So $Ly'' \leqslant_m H(f(y))''$ via $\{h(S_2^1(k_0, y))\}$ for some recursive function h.

$$(z, e) \notin L_x \leftrightarrow g_1(y, z, e) \in Ly''$$
$$\leftrightarrow \{h(S_2^1(k_0, y))\}(g_1(y, z, e)) \in H(f(y))''.$$

Define $\phi(x, z, e) \simeq \{h(S_2^1(k_0, y))\}(g_1(y, z, e))$, where $x = 2^y$.

Case 3. $x = 3 \cdot 5^y$.

$$L_x(z, e) \equiv (\exists u)(u <_e z) \;\&$$
$$\& \; (u)(u <_e z \to (k)(\exists v)(L_{\phi_y(k)}(v, e) \;\&\; u <_e v <_e z)$$
$$\equiv (\exists u)(u <_e z) \;\&$$
$$\& \; (u)(u <_e z \to (k)(\exists v)$$
$$\times (\{k_0\}(\phi_y(k), v, e) \in \overline{H(f(\phi_y(k))} \;\&$$
$$\& \; u <_e v <_e z).$$

Let

$$P(z, e) \equiv (\exists v)(\{k_0\}(\phi_y(k), v, e) \in \overline{H(f(\phi_y(k))} \;\&\; u <_e v <_e z).$$

Since it is Σ_1 in $H(f(\phi_y(k)))$ uniformly in y, we have $P(z, e) \leqslant_m H(2^{f(\phi_y(k))})$ via a partial recursive function $g(y, k, z, e)$ which is defined whenever $3 \cdot 5^y \in O$.

$$P(z, e) \leftrightarrow g(y, k, z, e) \in H(2^{f(\phi_y(k))})$$
$$\leftrightarrow \langle g(y, k, z, e), 2^{f(\phi_y(k))} \rangle \in H(x).$$

Hence $p \leqslant_m H(x)$. Now L_x is π_1 in P (uniformly in x). $\neg L_x (\leqslant_m p') \leqslant_m H'(x)$ via a recursive function $\lambda z, e\; h_1\,(x, z, e)$ where h_1 is a partial recursive function which is defined whenever $x = 3 \cdot 5^y \in O$. Define $\phi(x, z, e) \simeq h_1(x, z, e)$.

The above lemma shows L_x is $\pi_{r(|x|)}$.

Theorem 11.2. $W(\omega^{|x|}+1)\leqslant_m \overline{H'(f(x))}\ \forall x\in O-\{1\}$.

Proof. If $x=2$:

$$e\in W(\omega+1)\equiv e\in L\ \&\ (x)(\exists y)(z)(z<_e x\to z\leqslant y).$$

Now let $x\in O-\{1,2\}$.

$$e\in W(\omega^{|x|}+1)\equiv e\in L\ \&\ (x)(x\leqslant_e x\to \delta_d(e,x)\in W(\omega^{|x|})),$$

which is π_1 in $H(f(x))$.

Theorem 11.3. $\overline{H'(f(x))}\leqslant_m W(\sigma)$ *for any* σ *such that* $\omega^{|x|}<\sigma<\omega^{|x|+1}$, *where* $x\in O-\{1\}$.

Proof. Case 1. $|x|$ is finite. As in [3].

Case 2.

$$x=\underbrace{2^{\cdot^{\cdot^{2^{3\cdot 5^y}}}}}_{n},\quad n\geqslant 0.$$

$\overline{H'(f(x))}$ is of the form

$$\begin{aligned}&(x_1)(\exists y_1)(x_2)(\exists y_2)\cdots(x_n)(\exists y_n)(x_{n+1})(\exists y_{n+1})R^{H(3\cdot 5^y)}\\ &\equiv\neg(\exists x_1)(y_1)(\exists x_2)(y_2)\cdots(\exists x_n)(y_n)(\exists x_{n+1})(y_{n+1})\neg R^{H(3\cdot 5^y)}\\ &\equiv\neg(\exists x_1)(Uz_1)(Uz_2)\cdots(Uz_n)(y_{n+1})S^{H(3\cdot 5^y)}\end{aligned}$$

where $R^{H(3\cdot 5^y)}$, $S^{H(3\cdot 5^y)}$ are recursive in $H(3\cdot 5^y)$.

Now

$$\neg\forall y_{n+1}S^{H(3\cdot 5^y)}(\leqslant_m H'(3\cdot 5^y))\leqslant_m W(\omega^{|3\cdot 5^y|})\text{ nicely}$$

via a recursive function h. Choose p so that $\omega^{|x|}\cdot(p+1)<\sigma\leqslant\omega^{|x|}\cdot(p+2)$. For each x_1, let A_{x_1} be the well-ordered set described below:

$$\begin{aligned}&\langle z_1,\ldots,z_n,z\rangle\leqslant\langle z_1',\ldots,z_n',z'\rangle\quad\text{iff}\\ &\quad(\exists t)\leqslant_n[(i)<_t(z_i=z_i')\ \&\ z_t<z_t']\quad\text{or}\\ &\quad[(i)\leqslant_n(z_i=z_i')\ \&\ z\leqslant_{h(x_1,z_1,\ldots,z_n,u)}z']\end{aligned}$$

Then

$$\begin{aligned}&|A_{x_1}|=\omega^{|3\cdot 5^y|+n}=\omega^{|x|}\\ &\quad\text{if}\quad(Uz_1)\cdots(Uz_n)(y_{n+1})S^{H(3\cdot 5^y)}(x_1,z_1,\ldots,z_n,y_{n+1},n).\\ &|A_{x_1}|<\omega^{|x|}\quad\text{otherwise.}\end{aligned}$$

Let $C_{x_1}=A_{x_1},B_{x_1}$ (made disjoint), where

$$\begin{aligned}|B_{x_1}|&=\omega^{|x|-1}&&\text{if}\quad|x|\text{ is a successor,}\\ &=\omega^{\alpha_{x_1}}&&\text{if}\quad|x|\text{ is lim}\ \&\ \alpha_{x_1}\to|x|\text{ as }x_1\to\infty.\end{aligned}$$

Let C be such that $|C| = \omega^{|x|} \cdot p$, $A_u = C_0, C_1, C_2, \ldots$. There is a recursive function $h(u)$ such that $\leqslant_{h(u)} = A_u$.

$$u \in \overline{H'(f(x))} \to \begin{array}{lll} \omega^{|x|-1} \leqslant |C_{x_1}| < \omega^{|x|} & \forall x_1 & \text{if } |x| \text{ is a successor,} \\ \omega^{\alpha_{x_1}} \leqslant |C_{x_1}| < \omega^{|x|} & \forall x_1 & \text{if } |x| \text{ is limit} \end{array}$$
$$\to |\Sigma C_i| = \omega^{|x|}$$
$$\to |A_u| = \omega^{|x|} \cdot (p+1)$$
$$\to h(u) \in W(\sigma).$$

$$u \notin \overline{H'(f(x))} \to \exists x_1 |C_{x_1}| = \omega^{|x|}$$
$$\to |\Sigma C_i| \geqslant \omega^{|x|} \cdot 2$$
$$\to |A_u| \geqslant \omega^{|x|} \cdot (p+2) \geqslant \sigma$$
$$\to h(u) \notin W(\sigma).$$

Thus, $H'(f(x)) \leqslant_m W(\sigma)$.

Theorem 11.4. *For $\omega^{|x|} < \sigma < \omega^{|x|+1}$, $x \in O - \{1\}$, $W(\sigma) \leqslant_T H'(f(x))$.*

Proof. For each p, define

$$L_x^p(e) \leftrightarrow (\exists z_1) \cdots (\exists z_p)\Big(\bigwedge_{1 \leqslant i \leqslant p} L_x(z_i, e) \wedge \bigwedge_{1 \leqslant i < j \leqslant p} z_i \neq z_j \Big).$$

Since L_x is $\pi_{r(|x|)}$, L_x^p is $\Sigma_{r(|x|)+1}$. We now prove the theorem by induction on $|x|$.

Let $\sigma = \omega^{|x|} \cdot p + \tau$ where $2 \leqslant \tau \leqslant \omega^{|x|} + 1$. We proceed by induction on p. $p = 0$, $\sigma = \omega^{|x|} + 1$, $W(\omega^{|x|} + 1) \leqslant_m \overline{H'(f(x))}$, so $W(\omega^{|x|} + 1) \leqslant_T H'(f(x))$. We have

$$\begin{aligned} & e \in W \;\&\; \omega^{|x|} \cdot p + 1 \leqslant |e| < \sigma \quad \text{iff} \\ & \quad e \in L \;\&\; L_x^p(e) \;\&\; (x_1) \cdots (x_p)(x_1 <_e \cdots <_e x_p \;\& \\ & \quad L_x(x_1, e) \;\& \cdots \&\; L_x(x_p, e) \to \delta_d(e, x_1) \in W(\omega^{|x|} + 1) \\ & \quad \&\; \delta_b(e, x_1, x_2) \in W(\omega^{|x|} + 1) \;\& \cdots \&\; \delta_b(e, x_{p-1}, x_p) \in W(\omega^{|x|} + 1) \\ & \quad \&\; \delta_u(e, x_p) \in W(\tau))]. \end{aligned}$$

Since both $W(\tau)$ and $W(\omega^{|x|} + 1)$ are m-reducible to $\overline{H'(f(x))}$,

$$[e \in W \;\&\; \omega^{|x|} \cdot p + 1 \leqslant |e| < \sigma] \leqslant_T H'(f(x)).$$

Now

$$e \in W(\sigma) \quad \text{iff} \quad e \in W(\omega^{|x|} \cdot p + 1) \vee [e \in W \;\&\; \omega^{|x|} \cdot p + 1 \leqslant |e| < \sigma].$$

So $W(\sigma) \leqslant_T H'(f(x))$ by induction hypothesis and the above observation.

Theorem 11.5. *$W(\sigma) \leq_m \overline{H'(f(x))}$ for $\omega^{|x|} < \sigma < \omega^{|x|} \cdot 2$.*

Proof. Let $\sigma = \omega^{|x|} + \omega^{|y|} \cdot k + \tau$ where $y <_0 x$, $2 \leq \tau \leq \omega^{|y|} + 1$. We prove the theorem by induction on k.

$$\begin{aligned} e \in W(\sigma) \quad \text{iff} \quad & e \in W(\omega^{|x|} + \omega^{|y|} \cdot k + 1) \vee \\ & \{e \in L \ \& \ (z)(L_y(z, e) \to \delta_d(e, z) \in W(\omega^{|x|} + \omega^{|y|} \cdot k + 1)) \ \& \\ & (\exists z)(L_y(z, e) \ \& \ \delta_u(e, z) \in W(\tau))\}. \end{aligned}$$

The desired conclusion follows from Lemma 11.1, Theorems 11.2, 11.4 and the induction hypothesis.

By Theorems 11.3 and 11.5, $\deg_m W(\sigma) = \deg_m \overline{H'(f(x))}$ for $\omega^{|x|} < \sigma < \omega^{|x|} \cdot 2$ $(x \in O - \{1\})$; i.e. $\deg_m W(\sigma) = a_{r(\beta)+1}$ for $\omega^\beta < \sigma < \omega^\beta \cdot 2$ $(1 \leq \beta < \omega_1)$. We define h_α to be the degree of $H(x)$ where $|x| = \alpha$. By Theorems 11.3 and 11.4, $\deg W(\sigma) = \deg H'(f(x))$ for $\omega^{|x|} < \sigma < \omega^{|x|+1}$ $(x \in O - \{1\})$, i.e.

$$\deg W(\sigma) = \begin{cases} h_{2n+1} & \text{for} \quad \omega^n < \sigma < \omega^{n+1},\ n < \omega. \\ h_{r(\beta)+2} & \text{for} \quad \omega^\beta < \sigma < \omega^{\beta+1},\ \omega \leq \beta < \omega_1. \end{cases}$$

Since we know $\deg_m W(\omega^\beta)$, we also know their Turing's degrees.

$$\begin{aligned} &\deg W(\omega^n) = h_{2n} && 1 \leq n < \omega,^{6} \\ &\deg W(\omega^\beta) = h_{r(\beta)+1} && \omega \leq \beta < \omega_1. \end{aligned}$$

12. $W(\alpha)$ in hyperarithmetical difference hierarchy

We use the notations of Section 2 of [1]. Define $\Sigma_{\alpha,k} = \mathscr{D}_k(\Sigma^0_\alpha)$, $\pi_{\alpha,k} = \mathscr{D}_k(\pi^0_\alpha)$. Let $e_{\alpha,k}$ (resp. $a_{\alpha,k}$) denote the many-one degree of complete $\Sigma_{\alpha,k}$ (resp. $\pi_{\alpha,k}$) sets. We note the following partial ordering of m-degrees:

$$\left.\begin{matrix} e_\beta = e_{\beta,1} \\ a_\beta = a_{\beta,1} \end{matrix}\right\} < a_{\beta,2} = e_{\beta,2} < \left\{\begin{matrix} e_{\beta,3} \\ a_{\beta,3} \end{matrix}\right\} < a_{\beta,4} = e_{\beta,4} < \left\{\begin{matrix} e_{\beta,5} \\ a_{\beta,5} \end{matrix}\right\} <$$

$$\cdots < \begin{cases} e_{\beta+1} = e_{\beta+1,1}, \\ a_{\beta+1} = a_{\beta+1,1}. \end{cases}$$

The following two theorems are direct generalizations of Propositions 6.2 and 6.3 of [1].

Theorem 12.1. *For $p < \omega$, $1 \leq \beta < \omega_1$, $W(\omega^\beta \cdot (p+1) + 1)$ is $\pi_{r(\beta)+1,2p+1}$.*

[6] The case $n = 1$ is from $\deg_m W(\omega) = e_2 \wedge a_2$; cf. [1].

Proof. Let $|x| = \beta$. We show that $e \in W(\omega^\beta \cdot (p+1)+1)$ is of the form

$$(\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}) \cdot p \vee \pi_{r(\beta)+1}$$

by induction on p.

$$\begin{aligned}
e \in W(\omega^\beta \cdot (P+1)+1) &\leftrightarrow e \in W(\omega^\beta \cdot P+1) \vee \\
&\leftrightarrow [e \in W \ \&\ \omega^\beta \cdot p+1 \leqslant |e| \leqslant \omega^\beta \cdot (p+1)] \\
&\quad [e \in W \ \&\ \omega^\beta \cdot p+1 \leqslant |e| \leqslant \omega^\beta \cdot (p+1)] \\
&\leftrightarrow e \in L \ \&\ L_x^p(e) \ \& \\
&\quad \& \ (z_1) \cdots (z_p)\Big[\Big(\bigwedge_{1 \leqslant i < p} z_i <_e z_{i+1} \bigwedge_{1 \leqslant i \leqslant p} L_x(z_i, e)\Big) \\
&\rightarrow \delta_d(e, z_1) \in W(\omega^\beta + 1) \wedge \delta_u(e, z_p) \in W(\beta^\beta + 1) \ \& \\
&\quad \& \bigwedge_{1 \leqslant i < p} \delta_b(e, z_i, z_{i+1}) \in W(\omega^\beta + 1)\Big)\Big].
\end{aligned}$$

We note that $W(\omega^\beta)$ is $\Sigma_{r(\beta)}$. $W(\omega^{\beta+1})$ is $\pi_{r(\beta)+1}$, L is π_2, L_x is $\pi_{r(\beta)}$, L_x^p is $\Sigma_{r(\beta)+1}$. So

$$[e \in W \ \&\ \omega^\beta \cdot p+1 \leqslant |e| \leqslant \omega^\beta \cdot (p+1)]$$

is $\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}$. For $p=0$, $w(\omega^\beta + 1)$ is $\pi_{r(\beta)+1}$. Now, using the induction hypothesis $W(\omega^\beta \cdot (p+1)+1)$ is

$$(\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}) \cdot (p-1) \vee \pi_{r(\beta)-1} \vee (\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}),$$

i.e., $(\Sigma_{r(\beta)+1}) \cdot p \vee \pi_{r(\beta)+1}$.

Theorem 12.2. $1 \leqslant p < \omega$, $1 \leqslant \beta < \omega_1$, $W(\omega^\beta \cdot (p+1))$ *is* $\pi_{r(\beta)+1,2p}$.

Proof. We show that $e \in W(\omega^\beta \cdot (p+1))$ is of the form $(\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}) \cdot p$ by induction on p.

Let $\sigma(e)$ be a recursive function such that $e \in W$ iff $\sigma(e) \in W$ and in that case, $|\sigma(e)| = |e|+1$. For all $p \geqslant 1$,

$$\begin{aligned}
e \in W(\omega^\beta \cdot (p+1)) &\leftrightarrow \sigma(e) \in W(\omega^\beta \cdot (p+1)) \\
&\leftrightarrow \sigma(e) \in W(\omega^\beta \cdot p) \vee E,
\end{aligned}$$

where

$$\begin{aligned}
E \equiv &[e \in W \wedge \omega^\beta \cdot p+1 \leqslant |\sigma(e)| < \omega^\beta \cdot (p+1)] \\
&\leftrightarrow e \in L \ \&\ L_x^p(\sigma(e)) \ \&\ (z_1) \cdots (z_p) \\
&\quad \Big(\bigwedge_{1 \leqslant i < p} z_i < z_{i+1} \wedge \bigwedge_{1 \leqslant i \leqslant p} L_x(z_i, \sigma(e))\Big) \\
&\quad \rightarrow (\delta_d(\sigma(e), z_1) \in W(\omega^\beta + 1) \ \&\ \delta_u(\sigma(e), z_p) \in W(\omega^\beta) \ \& \\
&\quad \& \bigwedge_{1 \leqslant i < p} \delta_b(\sigma(e), z_i, z_{i+1}) \in W(\omega^\beta + 1))].
\end{aligned}$$

So E is $\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}$. For $p=1$,

$$e \in W(\omega^\beta \cdot 2) \leftrightarrow \sigma(e) \in W(\omega^\beta) \vee$$
$$[e \in W \wedge \omega^\beta + 1 \leq |\sigma(e)| < \omega^\beta \cdot 2],$$

which is $\Sigma_{r(\beta)+1} \wedge \pi_{r(\beta)+1}$.

13. Relativization

For each set x of natural numbers, we define $H^x(y)$ as in [6] and use it to define Σ^x_β by simply replacing $H(y)$ by $H^x(y)$ in the definition of Σ^0_β.[7] We further define $e^x_\beta, a^x_\beta, \Sigma^x_{\beta,k}, \ldots,$ by modifying naturally the definitions of $e_\beta, a_\beta, \Sigma_{\beta,k}, \ldots$ respectively. Let $W^x(\alpha)$ be the relativization of $W(\alpha)$ as in [8]. Most theorems in this paper have their obvious relativized versions.

In order to classify $W(\omega^\beta \cdot (p+1)+\eta)$ for $0<\eta<\omega^\beta$ and $1 \leq p < \omega$, we need a few lemmas. First we prove

Lemma 13.1. $H^{H(x)}(y) \equiv_T H(x +_o y)$, $\forall x, y \in O$.

Proof. The case $x=1$ is obvious. Now suppose $x \neq 1$, we shall prove the Turing's equivalence by induction on $y \in O$. $y=1$ is trivial.

$y = 2^z$:

$$H^{H(x)}(2^z) = (H^{H(x)}(z))' \equiv_T H'(x +_o z) = H((x +_o z) +_o 1_o)$$
$$\equiv_T H(x +_o 2^z).$$

$y = 3 \cdot 5^z$:

$$\langle u, v\rangle \in H^{H(x)}(3 \cdot 5^z) \leftrightarrow v <_o 3 \cdot 5^z \ \& \ u \in H^{H(x)}(v),$$

which is recursive in $H(x +_o v)$, hence recursive in $H(x +_o 3 \cdot 5^z)$. On the other hand,

$$\langle u, v\rangle \in H(x +_o 3 \cdot 5^z) \leftrightarrow v <_o x +_o 3 \cdot 5^z \ \& \ u \in H(v)$$

which, fixing v, is recursive in $H(x +_o \phi_z)(k))$ for some k uniformly in v. Now

$$H(x +_o \phi_z(k)) \leq_T H^{H(x)}(\phi_z(k)) \leq_T H^{H(x)}(3 \cdot 5^z)$$

uniformly. Thus $H(x +_o 3 \cdot 5^z) \leq_T H^{H(x)}(3 \cdot 5^z)$.

As a direct consequence of Lemma 13.1,

$$H^{H(x)}(2^y) \equiv_m H(x +_o 2^y).$$

Lemma 13.2. *Let α be an infinite ordinal, $x \in O$ & $|x| = \alpha$. Then*

$$\Sigma^{H(x)}_\beta = \Sigma_{\alpha+\beta} \quad \text{if } \beta \text{ is infinite},$$
$$= \Sigma_{\alpha+\beta-1} \quad \text{if } \beta \text{ is finite and } \beta \geq 1.$$

[7] Although $\Delta^1_1 = \bigcup_{\beta<\omega_1} \pi^0_\beta$, $\Delta^{1,x}_1 \neq \bigcup_{\beta<\omega_1} \pi^x_\beta$ for some x as indicated in 16–87 of [6].

Proof. First let $\beta=\gamma+n=|z|+n$ where γ is limit, $n\in\omega$, $z\in O$.

$$\begin{aligned}\Sigma_{\beta}^{H(x)} &= \Sigma_{n+1}^{H^{H(x)}}(z)=\Sigma_{n+1}^{H(x+_O z)}=\Sigma_{|x+_O z|+n}\\ &= \Sigma_{|x|+|z|+n}=\Sigma_{\alpha+\beta}.\end{aligned}$$

For the second case, let $\alpha=\gamma+k=|3\cdot 5^y|+k$ where $3\cdot 5^y\in O$

$$\begin{aligned}\Sigma_{\beta}^{H(x)} &= \Sigma_{\beta}^{H^{(k)}(3\cdot 5^y)}=\Sigma_{k+\beta}^{H(3\cdot 5^y)}=\Sigma_{\gamma+(k+\beta)-1}\\ &= \Sigma_{\alpha+\beta-1}.\end{aligned}$$

Now, let e^x be the ordering $\{\langle u,v\rangle \mid L_x(u,e)\ \&\ L_x(v,e)\ \&\ u\leqslant_e v\}$. We shall generize Lemma 4.7 of [1][8] to the following lemma, the proof of which is a carry over with some routine modifications.

Lemma 13.3. *Let P be any predicate, $Q=H^P(f(x))$ $(x\in O-\{1\})$. Then for each $t\geqslant 1$, there is a recursive function $\delta_t(e)$ satisfying*

(i) *if $e\notin W^P$ then $\delta_t(e)\in L^Q-W^Q$,*
(ii) *if $e\in W^P$ then $\delta_t(e)\in W^Q$; and in this case,*
(iii) *if $|e^x|^P<\omega$, then $|\delta_t(e)|^Q<\omega$,*
(iv) *if $|e^x|^P\geqslant\omega$, then $|e^x|^P\leqslant|\delta_t(e)|^Q<|e^x|^P+\omega$,*
(v) *$|\delta_t(e)|^Q=|e^x|^P$ if $|e^x|^P=\omega\cdot\beta+k$, where $\beta\geqslant 1$ and $0\leqslant k\leqslant t$.*

Proof. Omit.

Corollary. $W^P(\omega^{|x|}\cdot(\omega\cdot\beta+k))\leqslant_m W^{H^P(f(x))}(\omega\cdot\beta+k)$*(via δ_k).*

Proof. Apply Lemma 13.3 to $\sigma(e)$.

Corollary. $W^P(\omega^{|x|}\cdot(\omega\cdot\beta+k)+1)\leqslant_m W^{H^P(f(x))}(\omega\cdot\beta+1)$.

Proof. $k=0$. Directly from Lemma 13.3

$k>0$. Lemma 13.3 gives

$$W^P(\omega^{|x|}\cdot(\omega\cdot\beta+k)+1)\leqslant_m W^{H^P(f(x))}(\omega\cdot\beta+k).$$

We note $W^A(\alpha+1)\equiv_m W^A(\alpha+k)$ for $0<k<\omega$.

Lemma 13.4. *For every limit ordinal η, there are unique τ,σ,k satisfying*

$$\eta=\omega^{\tau}\cdot(\omega\cdot\sigma+k)\ \&\ 0<k<\omega.$$

Proof. Let $P(r)$ be $\forall\beta(\eta\neq\omega^r\cdot\beta)$. Define $r_0=\mu rP(r)$. Then r_0 is a successor ordinal. Let $r_0=\tau+1$. Since $\neg P(\tau)$, $\exists\beta(\eta=\omega^\tau\cdot\beta)$. This β can be written as $\omega\cdot\sigma+k$, $k\neq 0$.

[8] Q'' in Lemma 4.7 of [1] should be the double jump of Q.

Theorem 13.5.

$$\deg_m W(\omega^\beta \cdot (p+1)+\eta) \leq a_{r(\beta)+1,2p+1},$$

where $1 \leq p < \omega$, $1 \leq \eta < \omega^\beta$.

Proof. We shall prove that

for any A, $W^A(\omega^\beta \cdot (p+1)+\eta)$ is of the form $(\Sigma^A_{r(\beta)+1} \wedge \pi^A_{r(\beta)+1}) \cdot p \vee \pi^A_{r(\beta)+1}$,

by induction on β.

The relativized version of Theorem 6.7 of [1] provides the case β is finite. Now suppose β is infinite. Assuming (*) holds for all $\beta_1 < \beta$, we show (*) holds for β by induction on η.

Base: $\eta = 1$. By the relativized version of Theorem 12.1.

Induction Step: (1) η is finite. Since $W^A(\beta^\beta \cdot (p+1)+\eta) \equiv_m W^A(\omega^\beta \cdot (p+1)+1)$, it can be reduced to the base case.

(2) η is limit. By Lemma 13.4 $\eta = \omega^\tau \cdot (\omega \cdot \sigma + k_1)$ where $0 < k_1 < \omega$ (thus, $\omega \cdot \sigma + k_1 < \eta$). Let $\beta = \tau + \beta_1$. From the first Corollary of Lemma 13.3,

$$W^A(\omega^\beta \cdot (p+1)+\eta) = W^A(\omega^\tau \cdot (\omega^{\beta_1} \cdot (p+1) + \\ +(\omega \cdot \sigma + k_1))) \leq_m W^{H^A(f(x))}(\omega^{\beta_1} \cdot (p+1)+(\omega \cdot \sigma + k_1)),$$

where $|x| = \tau$. By the induction hypothesis, the last set, hence the first set, is

$$(\Sigma^{H^A(f(x))}_{r(\beta_1)+1} \wedge \pi^{H^A(f(x))}_{r(\beta_1)+1}) \cdot p \vee \pi^{H^A(f(x))}_{r(\beta_1)+1}.$$

Now, if $r(\beta_1)$ is infinite

$$\Sigma^{H^A(f(x))}_{r(\beta_1)+1} = \Sigma^A_{r(\tau)+1+r(\beta_1)+1} = \Sigma^A_{r(\beta)+1}.$$ [9]

If $r(\beta_1)$ is finite,

$$\Sigma^{H^A(f(x))}_{r(\beta_1)+1} = \Sigma^A_{r(\tau)+1+r(\beta_1)+1-1} \\ = \Sigma^A_{r(\beta)+1}.$$

(3) $\eta = \eta_1 + k$ where $1 \leq k < \omega$ & η_1 limit. By Lemma 13.4, $\eta_1 = \omega^\tau \cdot (\omega \cdot \sigma + k_1)$, $k_1 > 0$. Let $\beta = \tau + \beta_1$.

$$W^A(\omega^\beta \cdot (p+1)+\eta) \leq_m W^A(\omega^\beta \cdot (p+1)+\eta_1+1) \\ = W^A(\omega^{\tau+\beta_1}(p+1)+\omega^\tau \cdot (\omega \cdot \sigma + k_1)+1) \\ = W^A(\omega^\tau \cdot [\omega^{\beta_1} \cdot (p+1)+(\omega \cdot \sigma + k_1)]+1) \\ \leq_m W^{H^A(f(x))}(\omega^{\beta_1} \cdot (p+1)+\omega \cdot \sigma + 1),$$

where $|x| = \tau$. By the induction hypothesis, it is

$$(\Sigma^{H^A(f(x))}_{r(\beta_1)+1} \wedge \pi^{H^A(f(x))}_{\alpha(\beta_1)+1}) \cdot p \vee \pi^{H^A(f(x))}_{r(\beta_1)+1}.$$

As in (2), it is of the right form.

[9] Note $r(\alpha + \beta) = r(\alpha) + r(\beta)$.

Theorem 13.6.

$$W^{H(f(x))}(\omega^{|y|}) \equiv_m W(\omega^{|x|+|y|}) \quad for \quad x \in O,\ y \in O-\{1,2\}.$$

Proof.

$$\begin{aligned} W^{H(f(x))}(\omega^{|y|}) &\equiv_m H^{H(f(x))}(f(y)) \equiv_m H(f(x)+_o f(y)) \\ &\equiv_m H(f(x+_o y)) \equiv_m W(\omega^{|x+_o y|}) \\ &\equiv_m W(\omega^{|x|+|y|}).^{10} \end{aligned}$$

Theorem 13.7.

$$W^{H(f(x))}(\omega^{|y|}+1) \equiv_m W(\omega^{|x|+|y|}+1) \quad for \quad x \in O,\ y \in O-\{1\}.$$

Proof.

$$\begin{aligned} W^{H(f(x))}(\omega^{|y|}+1) &\equiv_m \overline{H^{H(f(x))}(f(y)+_o 1_o)} \\ &\equiv_m \overline{H(f(x)+_o(f(y)+_o 1_o))} \\ &\equiv_m \overline{H(f(x+_o y)+_o 1_o)} \\ &\equiv_m W(\omega^{|x|+|y|}+1). \end{aligned}$$

Similarly, we have

$$\mathrm{WF}^{H(f(x))}(\omega\cdot|y|) \equiv_m \mathrm{WF}(\omega\cdot(|x|+|y|)) \quad \forall x \in O,\ y \in O-\{1,2\},$$
$$\mathrm{WF}^{H(f(x))}(\omega\cdot|y|+1) \equiv_m \mathrm{WF}(\omega\cdot(|x|+|y|)+1) \quad \forall x \in O,\ y \in O-\{1\}.$$

14. Conclusion

Using the results of Section 3 of [3], we can determine the many-one degrees of $O(\alpha)$'s[11] from the knowledge of $\deg_m W(\alpha)$'s. We summarize all results known on the many-one degrees of $\mathrm{WF}(\alpha)$, $W(\alpha)$ and $O(\alpha)$ in the Tables 3–5. We hope that the "$\leq$" in Tables 4 and 5 will be dropped soon.

Table 3

α	$\deg_m \mathrm{WF}(\alpha)$
0	$\deg_m \phi$
$1 \leq \alpha < \omega$	a_2
ω	$a_{2,2}$
$\omega\cdot\beta \quad (1<\beta<\omega_1)$	$e_{r(\beta)}$[a]
$\omega\cdot\beta+k \quad (1\leq\beta<\omega_1,\ 0<k<\omega)$	$a_{r(\beta)+1}$

[a] Recall $r(\omega\cdot\beta+n)=\omega\cdot\beta+2n,\ n<\omega$.

[10] $|f(x+_o y)|=|f(x)|+|f(y)|$.

[11] $O(\alpha)=\{x\in O \mid |x|<\alpha\}$.

Table 4

α	$\deg_m W(\alpha)$
0	$\deg_m \phi$
$1 \leq \alpha < \omega$	a_2
ω	$a_{2,2}$
$\omega^\beta \quad (1 < \beta < \omega_1)$	$e_{r(\beta)}$
$\omega^\beta < \alpha < \omega^\beta \cdot 2 \quad (1 \leq \beta < \omega_1)$	$a_{r(\beta)+1}$
$\omega^n \cdot (p+1) \quad (1 \leq n < \omega, 1 \leq p < \omega)$	$a_{2n+1,2p}$
$\omega^n \cdot (p+1) < \alpha < \omega^n \cdot (p+2) \quad (1 \leq n < \omega, 1 \leq p < \omega)$	$a_{2n+1,2p+1}$
$\omega^\beta \cdot (p+1) \quad (\omega \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p}$
$\omega^\beta \cdot (p+1) < \alpha < \omega^\beta \cdot (p+2) \quad (\omega \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p+1}$

Table 5

α	$\deg_m O(\alpha)$
0	$\deg_m \phi$
$0 < \alpha \leq \omega$	$\deg_m (\{0\})$
$\omega < \alpha < \omega^2$	a_2
ω^2	$a_{2,2}$
$\omega^{n+1} \quad (1 < n < \omega)$	e_{2n}
$\omega^{n+1} \cdot (p+1) \quad (1 \leq n < \omega, 1 \leq p < \omega)$	$a_{2n+1,2p}$
$\omega^{n+1} \cdot (p+1) < \alpha < \omega^{n+1} \cdot (p+2) \quad (1 \leq n < \omega, 0 \leq p < \omega)$	$a_{2n+1,2p+1}$
$\omega^\beta \quad (\omega \leq \beta < \omega_1)$	$e_{r(\beta)}$
$\omega^\beta < \alpha < \omega^\beta \cdot 2 \quad (\omega \leq \beta < \omega_1)$	$a_{r(\beta)+1}$
$\omega^\beta \cdot (p+1) \quad (\omega \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p}$
$\omega^\beta \cdot (p+1) < \alpha < \omega^\beta \cdot (p+2) \quad (\omega \leq \beta < \omega_1, 1 \leq p < \omega)$	$\leq a_{r(\beta)+1,2p+1}$

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References

[1] L. Hay, A. Manaster and J. Rosenstein, Small Recursive Ordinals, Many-one degrees, and the Arithmetical Difference Hierarchy, Ann. Math. Logic 8 (1975) 297–343.

[2] S.C. Kleene, On the forms of the predicates in the theory of construction ordinals, Am. J. Math. 77 (1955) 405–428.

[3] G. Kreisel, J. Shoenfield and H. Wang, Number theoretic concepts and recursive well-orderings, Arch. Math. Logik Grundlagenforsch. 5 (1960) 42–64.

[4] S.C. Liu, A note on many-one reducibility, J. Symbolic Logic 28 (1963) 35–42.

[5] S.C. Liu, On many-one degrees, J. Symbolic Logic 28 (1963) 143–153.

[6] H. Rogers, Jr., Theory of recursive functions and effective computability (McGraw-Hill, New York, 1967).

[7] J. Shoenfield, Mathematical Logic (Addison-Wesley, Reading, MA, 1967).

[8] C. Spector, Recursive Well-orderings, J. Symbolic Logic 20 (1955) 151–163.

[9] C. Spector, Strongly invariant hierarchies, Abstract, Notices Am. Math. Soc. 5 (1958) 851.